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An integral structure in quantum cohomology and mirror symmetry for toric orbifolds

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Abstract

We introduce an integral structure in orbifold quantum cohomology associated to the K-group and the $\widehat{\Gamma}$-class. In the case of compact toric orbifolds, we show that this integral structure matches with the natural integral structure for the Landau–Ginzburg model under mirror symmetry. By assuming the existence of an integral structure, we give a natural explanation for the specialization to a root of unity in Y. Ruan's crepant resolution conjecture [Yongbin Ruan, The cohomology ring of crepant resolutions of orbifolds, in: Contemp. Math., vol. 403, Amer. Math. Soc., Providence, RI, 2006, pp. 117–126].

Keywords: Quantum cohomology; Variation of Hodge structures; Semi-infinite variation of Hodge structures; Mirror symmetry; Landau–Ginzburg model; Toric Deligne–Mumford stack; Orbifold; Orbifold quantum cohomology; Crepant resolution conjecture; Ruan's conjecture; K-theory; McKay correspondence; Oscillatory integral; Hypergeometric function; GKZ-system; Singularity theory; Gamma class

Contents

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1. Introduction

Mirror symmetry for Calabi–Yau manifolds can be formulated as an isomorphism of *variations of Hodge structures* (VHS for short): The A-model VHS [60] defined by the genus zero Gromov–Witten theory of X is isomorphic to the B-model VHS associated to deformation of complex structures of the mirror Y. As a consequence, one can calculate Gromov–Witten invariants of X from Picard–Fuchs equations for Y; such phenomena have been checked in many examples including toric complete intersections [26,38]. However, while the B-model VHS has a natural integral local system $H^n(Y,\mathbb{Z})$, the A-model VHS seems to lack an integral structure. In this paper, we study the question *"What is the integral local system in the A-model mirrored from the B-model?"* Our calculation on compact toric orbifolds suggests that the K-group of X should give the integral local system in the A-model.

Let us describe our K-theory integral structure in the A-model. The genus zero Gromov–Witten theory defines a family of commutative algebras $(H^*(X),\circ_\tau)$ on the cohomology group parametrized by $\tau\in H^*(X)$, called *quantum cohomology*. The *quantum D-module* is given by a flat connection ∇ on the trivial bundle $H^*(X)\times H^*(X)\to H^*(X)$ with a parameter $z\in\mathbb{C}^*$, called the *Dubrovin connection*:

$$\nabla_X = d_X + \frac{1}{z}X\circ_\tau,\quad X\in H^*(X),$$

where τ denotes a point on the base and d_X is the directional derivative (with respect to the given trivialization). We can extend the Dubrovin connection in the direction of the parameter z (see Definition 2.2) and get a flat $H^*(X)$-bundle over $H^*(X)\times\mathbb{C}^*$. A general solution to the differential equation $\nabla s(\tau,z)=0$ is of the form $s(\tau,z)=L(\tau,z)z^{-\mu}z^{c_1(X)}\phi$ for some $\phi\in H^*(X)$. Here μ is the grading operator (7) and $L(\tau,z)$ is the fundamental solution (11) which is asymptotic to $e^{-\tau/z}$ in the large radius limit (5). Let $\delta_1,\dots,\delta_n$ be the Chern roots of the tangent bundle TX and define a transcendental characteristic class $\widehat{\Gamma}(T\mathcal{X})$ by (see (23) for orbifold case)

$$\widehat{\Gamma}(TX) := \prod_{i=1}^n \Gamma(1+\delta_i) = \exp\biggl(-\gamma c_1(X) + \sum_{k\geqslant 2}(-1)^k(k-1)!\zeta(k)\operatorname{ch}_k(TX)\biggr),$$

where γ is the Euler constant and $\zeta(s)$ is the Riemann zeta function. For $V\in K(X)$, we define a ∇-flat section $\mathcal{Z}(V)$ to be (see (24))

$$\mathcal{Z}(V)(\tau,z) := (2\pi)^{-n/2}L(\tau,z)z^{-\mu}z^{c_1(X)}\bigl(\widehat{\Gamma}(TX)\cup(2\pi\mathbf{i})^{\deg/2}\operatorname{ch}(V)\bigr),$$

where $n=\dim X$. These flat sections $\mathcal{Z}(V)$, $V\in K(X)$ define an integral lattice in the space of ∇-flat sections. We call it the *$\widehat{\Gamma}$-integral structure*.

The mirror of a compact toric orbifold is given by a Landau–Ginzburg (LG) model. It is a pair of a torus $Y_q=(\mathbb{C}^*)^n$ and a Laurent polynomial $W_q: Y_q\to\mathbb{C}$ on it (q is a parameter). The LG model defines a *B-model D-module* which is underlain by a natural integral local system generated by *Lefschetz thimbles* of W_q. Under mirror symmetry (Conjecture 4.3), the quantum D-module of a toric orbifold is isomorphic to the B-model D-module (Proposition 4.8). Our main theorem is the following:

Theorem 1.1. *(See Theorem 4.11.) Let $\mathcal{X}$ be a weak Fano projective toric orbifold constructed from the initial data satisfying $\hat{\rho} \in \widetilde{C}_{\mathcal{X}}$ (see Section* 3.1*). Assume that mirror theorem (Conjecture* 4.3*) and Assumption* 2.7(c) *hold for $\mathcal{X}$. The $\widehat{\Gamma}$-integral structure on the quantum D-module corresponds to the natural integral local system of the B-model D-module under the mirror isomorphism in Proposition* 4.8.

Conjecture 4.3 will be proved in joint work [25] with Coates, Corti and Tseng. In fact, both of the assumptions in the theorem are known to be true for toric *manifolds*. This theorem follows from the following equality of "central charges." We define the *quantum cohomology central charge* of $V \in K(X)$ to be

$$Z(V)(\tau, z) := \frac{(2\pi z)^{n/2}}{(2\pi \mathbf{i})^n} \int_X \mathcal{Z}(V)(\tau, z).$$

Under Conjecture 4.3, $Z(V)$ is given as a pairing of $\mathrm{ch}(V)$ and a cohomology-valued hypergeometric series $H(q, z)$ (see (34) and (73)).

Theorem 1.2. *(See Theorem 4.14.) Under the same assumptions as Theorem* 1.1*, the quantum cohomology central charge of the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is given by the oscillatory integral over the real Lefschetz thimble $\Gamma_{\mathbb{R}}$:*

$$Z(\mathcal{O}_{\mathcal{X}})\big(\tau(q), z\big) = \frac{1}{(2\pi \mathbf{i})^n} \int_{\Gamma_{\mathbb{R}} \subset Y_q} e^{-W_q(y)/z} \omega_q \tag{1}$$

where $\tau = \tau(q)$ is a mirror map in Lemma 4.2.

The relationship between K-theory and quantum cohomology can be foreseen by Kontsevich's *homological mirror symmetry*. The integral local system of the B-model VHS on a Calabi–Yau Y can be measured by integration (period) over a Lagrangian n-cycle, an object of the Fukaya category of Y (A-type D-brane). Therefore, by homological mirror symmetry, a coherent sheaf on X, an object of the derived category of X (B-type D-brane) should have a pairing with the quantum D-module and give a (dual) flat section of the Dubrovin connection. The quantum cohomology central charge $Z(V)$ can be viewed as a "period of V" and the equality (1) should be generalized as

$$Z(V)\big(\tau(q), z\big) = \frac{1}{(2\pi \mathbf{i})^n} \int_{\mathrm{mir}(V)} e^{-W_q/z} \omega_q,$$

where $\mathrm{mir}(V)$ is the Lefschetz thimble mirror to V. Theorem 1.1 shows the existence of the map $V \mapsto \mathrm{mir}(V)$ on the K-group level. This shows a K-group version of Dubrovin's conjecture (Corollary 4.12).

In the context of toric mirror symmetry, closely related observations have been made by Horja [44], Hosono [45] and Borisov and Horja [11]. Borisov and Horja [11] identified the space of solutions to the GKZ-system (corresponding to a toric Calabi–Yau $\mathcal{X}$) with the K-group of $\mathcal{X}$ and showed that the analytic continuation of a solution corresponds to a Fourier–Mukai transformation between birational $\mathcal{X}$'s. Hosono [45] proposed a central charge formula for Calabi–Yau complete intersections in toric varieties in terms of an explicit hypergeometric series. Our observation is based on non-Calabi–Yau examples, but all of their results can be understood from the $\widehat{\Gamma}$-integral structure. After the preprint version [49] of this paper was written, a rational structure based on the same $\widehat{\Gamma}$-class was proposed by Katzarkov, Kontsevich, and Pantev [53] independently.

We hope that an integral structure exists globally on the Kähler moduli space — the (maximal) base space where the quantum cohomology is analytically continued. A global existence of an integral structure is relevant to Yongbin Ruan's *crepant resolution conjecture*. Roughly speaking, it says that for a crepant resolution Y of an orbifold $\mathcal{X}$, quantum cohomology of $\mathcal{X}$ and Y are related by analytic continuation. In joint work [27] with Coates and Tseng, we proposed the picture that the *semi-infinite variations of Hodge structures* ($\frac{\infty}{2}$VHS) associated to quantum cohomology of $\mathcal{X}$ and Y match under a linear symplectic transformation $\mathbb{U}: \mathcal{H}^{\mathcal{X}} \to \mathcal{H}^{Y}$ between Givental's symplectic spaces $\mathcal{H}^{\mathcal{X}}$, $\mathcal{H}^{Y}$ (which are loop spaces on cohomology groups, see (26)). This implies that the quantum D-modules of $\mathcal{X}$ and Y are isomorphic after analytic continuation. In this paper, we furthermore conjecture that *the isomorphism of quantum D-modules preserves the K-theory integral structures on the both sides*. Then the symplectic transformation $\mathbb{U}$ would be induced from an isomorphism $\mathbb{U}_K : K(\mathcal{X}) \to K(Y)$ of K-groups (93) (Ψ below involves the $\widehat{\Gamma}$ class, see (24)):

$$\begin{tikzcd}
K(\mathcal{X}) \arrow[r, "\mathbb{U}_K"] \arrow[d, "z^{-\mu}z^{\rho}\Psi^{\mathcal{X}}"'] & K(Y) \arrow[d, "z^{-\mu}z^{\rho}\Psi^{Y}"] \\
\mathcal{H}^{\mathcal{X}} \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}(\widetilde{\mathbb{C}^*}) \arrow[r, "\mathbb{U}"] & \mathcal{H}^{Y} \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}(\widetilde{\mathbb{C}^*}).
\end{tikzcd}$$

In view of Borisov and Horja [11], we hope that $\mathbb{U}_K$ is given by a geometric correspondence such as a Fourier–Mukai transformation. This picture (Proposal 5.7) gives us a natural reason why the quantum parameters should be specialized to roots of unity at the orbifold large radius limit point. In some cases, one can predict explicitly the specialization value/co-ordinate change using the $\widehat{\Gamma}$-class.

This paper is a revision of the preprint [49], where we also studied possible *real structures* on quantum cohomology $\frac{\infty}{2}$VHS, yielding Hertling's TERP structure [41,42]. We showed that the (p, p)-part of quantum cohomology $\frac{\infty}{2}$VHS is *pure* and *polarized* near the large radius limit point with respect to the real structure induced from the $\widehat{\Gamma}$-integral structure [49, Theorem 3.7]. These properties — pure and polarized — are semi-infinite analogues of the Hodge decomposition and Hodge–Riemann bilinear inequality and yield *tt^*-geometry* [17,41] on quantum cohomology. The real structure part of the preprint [49] will appear in a separate paper [51].

The paper is organized as follows. In Section 2, we introduce the $\widehat{\Gamma}$-integral structure in orbifold quantum cohomology after reviewing the basics on orbifold quantum cohomology. In Section 3, we introduce Landau–Ginzburg mirrors to toric orbifolds and construct the B-model D-module from the LG model. In Section 4, we formulate mirror symmetry for toric orbifolds in terms of a D-module, and prove the main theorem (Theorem 4.11). In Section 5, we propose the crepant resolution conjecture with an integral structure (Proposal 5.7) and study specialization values of quantum parameters using the notion of integral periods.

We assume the convergence of quantum cohomology throughout the paper. We consider only the even parity part of the cohomology, i.e. $H^*(X)$ means $\bigoplus_k H^{2k}(X)$. We also assume that a smooth Deligne–Mumford stack $\mathcal{X}$ in this paper has the resolution property, i.e. every coherent sheaf is a quotient of a vector bundle, so that we can apply the orbifold Riemann–Roch (22) to $\mathcal{X}$. (A toric orbifold has this property. See [71, Theorem 2.1].) Note that the orbifold cohomology $H^*_{\mathrm{orb}}(\mathcal{X})$ is denoted also by $H^*_{\mathrm{CR}}(\mathcal{X})$ in the literature.

1.1. Notation

$\mathbf{i}$	imaginary unit $\mathbf{i}^2 = -1$;
$\mathcal{X}$	smooth Deligne–Mumford stack;
X	coarse moduli space of $\mathcal{X}$;
$I\mathcal{X}$	inertia stack of $\mathcal{X}$;
$\mathsf{T} = \{0\} \cup \mathsf{T}'$	index set of inertia components;
$\mathrm{inv}: I\mathcal{X} \to I\mathcal{X},\ \mathsf{T} \to \mathsf{T}$	involution $(x, g) \mapsto (x, g^{-1})$;
ι_v	age of inertia component $v \in \mathsf{T}$;
n, n_v	$\dim_{\mathbb{C}} \mathcal{X}$, $\dim_{\mathbb{C}} \mathcal{X}_v$.

2. Integral structure in quantum cohomology

In this section, we review orbifold quantum cohomology and introduce the integral structure associated to the K-group and the $\widehat{\Gamma}$-class. Gromov–Witten theory for orbifolds has been developed by Chen and Ruan [19,20] in the symplectic category and by Abramovich, Graber, and Vistoli [2] in the algebraic category. The definition of the integral structure makes sense for both categories, but we work in the algebraic category.

2.1. Orbifold quantum cohomology

Let $\mathcal{X}$ be a proper smooth Deligne–Mumford stack over $\mathbb{C}$. Let $I\mathcal{X}$ denote the *inertia stack* of $\mathcal{X}$, defined by the fiber product $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ of the two diagonal morphisms $\Delta: \mathcal{X} \to \mathcal{X} \times \mathcal{X}$. A point on $I\mathcal{X}$ is given by a pair (x, g) of a point $x \in \mathcal{X}$ and $g \in \mathrm{Aut}(x)$. We call g the *stabilizer* of $(x, g) \in I\mathcal{X}$. Let T be the index set of components of the $I\mathcal{X}$. Let $0 \in \mathsf{T}$ be the distinguished element corresponding to the trivial stabilizer. Set $\mathsf{T}' = \mathsf{T} \setminus \{0\}$. We have

$$I\mathcal{X} = \bigsqcup_{v\in\mathsf{T}} \mathcal{X}_v = \mathcal{X}_0 \cup \bigsqcup_{v\in\mathsf{T}'} \mathcal{X}_v, \quad \mathcal{X}_0 = \mathcal{X}.$$

We associate a rational number ι_v to each connected component $\mathcal{X}_v$ of $I\mathcal{X}$. This is called *age* or *degree shifting number*. Take a point $(x, g) \in \mathcal{X}_v$ and let

$$T_x\mathcal{X} = \bigoplus_{0\leqslant f<1} (T_x\mathcal{X})_f$$

be the eigenspace decomposition of $T_x\mathcal{X}$ with respect to the stabilizer action, where g acts on $(T_x\mathcal{X})_f$ by $\exp(2\pi\mathbf{i}f)$. We define

$$\iota_v = \sum_{0\leqslant f<1} f \dim_{\mathbb{C}}(T_x\mathcal{X})_f.$$

This is independent of the choice of a point $(x, g) \in \mathcal{X}_v$. The *(even parity) orbifold cohomology group* $H^*_{\mathrm{orb}}(\mathcal{X})$ is defined to be the sum of the (even degree) cohomology of $\mathcal{X}_v$, $v \in \mathsf{T}$:

$$H^k_{\mathrm{orb}}(\mathcal{X}) = \bigoplus_{\substack{v\in\mathsf{T}\\ k-2\iota_v\equiv 0(2)}} H^{k-2\iota_v}(\mathcal{X}_v, \mathbb{C}).$$

The degree k of the orbifold cohomology can be a fractional number in general. Each factor $H^*(\mathcal{X}_v, \mathbb{C})$ in the right-hand side is same as the cohomology group of $\mathcal{X}_v$ as a topological space. If not otherwise stated, we will use $\mathbb{C}$ as the coefficient of cohomology groups. We have an involution $\mathrm{inv}: I\mathcal{X} \to I\mathcal{X}$ defined by $\mathrm{inv}(x, g) = (x, g^{-1})$. This induces an involution $\mathrm{inv}: \mathsf{T} \to \mathsf{T}$. The *orbifold Poincaré pairing* is defined to be

$$(\alpha, \beta)_{\mathrm{orb}} := \int_{I\mathcal{X}} \alpha \cup \mathrm{inv}^*(\beta) = \sum_{v\in\mathsf{T}} \int_{\mathcal{X}_v} \alpha_v \cup \beta_{\mathrm{inv}(v)},$$

where α_v, β_v are the v-components of α, β. This pairing is symmetric, non-degenerate over $\mathbb{C}$ and of degree $-2n$, where $n = \dim_{\mathbb{C}} \mathcal{X}$.

Now we assume that the coarse moduli space X of $\mathcal{X}$ is projective. The *genus zero Gromov–Witten invariants* are integrals of the form:

$$\left\langle \alpha_1 \psi^{k_1}, \ldots, \alpha_l \psi^{k_l} \right\rangle^{\mathcal{X}}_{0,l,d} = \int_{[\mathcal{X}_{0,l,d}]^{\mathrm{vir}}} \prod_{i=1}^{l} \mathrm{ev}_i^*(\alpha_i) \psi_i^{k_i} \tag{2}$$

where $\alpha_i \in H^*_{\mathrm{orb}}(\mathcal{X})$, $d \in H_2(X, \mathbb{Z})$ and k_i is a non-negative integer. $[\mathcal{X}_{0,l,d}]^{\mathrm{vir}}$ is the virtual fundamental class of the moduli stack $\mathcal{X}_{0,l,d}$ of genus zero, l-pointed stable maps to $\mathcal{X}$ of degree d; $\mathrm{ev}_i: \mathcal{X}_{0,l,d} \to I\mathcal{X}$ is the evaluation map[1] at the ith marked point; ψ_i is the first Chern class of the line bundle over $\mathcal{X}_{0,l,d}$ whose fiber at a stable map is the cotangent space of the coarse curve at the ith marked point. (Our notation is taken from [26]; $\mathcal{X}_{0,l,d}$ is denoted by $\mathcal{K}_{0,l}(\mathcal{X}, d)$ in [2].) The correlator (2) is non-zero only when d belongs to $\mathrm{Eff}_{\mathcal{X}} \subset H_2(X, \mathbb{Z})$, the semigroup generated by effective stable maps, and $\sum_{i=1}^{l}(\deg \alpha_i + 2k_i) = 2n + 2\langle c_1(T\mathcal{X}), d\rangle + 2l - 6$.

Let $\{\phi_k\}_{k=1}^N$ and $\{\phi^k\}_{k=1}^N$ be bases of $H^*_{\mathrm{orb}}(\mathcal{X})$ which are dual with respect to the orbifold Poincaré pairing, i.e. $(\phi_i, \phi^j)_{\mathrm{orb}} = \delta_i^j$. The *orbifold quantum product* $\bullet_\tau$ is a formal family of commutative and associative products on $H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}[\![\mathrm{Eff}_{\mathcal{X}}]\!]$ parametrized by $\tau \in H^*_{\mathrm{orb}}(\mathcal{X})$. This is defined by the formula:

$$\alpha \bullet_\tau \beta = \sum_{d\in\mathrm{Eff}_{\mathcal{X}}} \sum_{l\geqslant 0} \sum_{k=1}^{N} \frac{1}{l!} \langle \alpha, \beta, \tau, \ldots, \tau, \phi_k \rangle^{\mathcal{X}}_{0,l+3,d} Q^d \phi^k,$$

[1] The map ev_i here is defined only as a map of topological spaces (not as a map of stacks). The evaluation map defined in [2] is a map of stacks but takes values in the *rigidified inertia stack*, which is the same as $I\mathcal{X}$ as a topological space but is different as a stack.

where Q^d is the element of the group ring $\mathbb{C}[\mathrm{Eff}_{\mathcal{X}}]$ corresponding to $d \in \mathrm{Eff}_{\mathcal{X}}$. We decompose $\tau \in H^*_{\mathrm{orb}}(\mathcal{X})$ as

$$\tau = \tau_{0,2} + \tau', \qquad \tau_{0,2} \in H^2(\mathcal{X}), \qquad \tau' \in \bigoplus_{k \neq 1} H^{2k}(\mathcal{X}) \oplus \bigoplus_{v \in \mathsf{T}'} H^*(\mathcal{X}_v). \tag{3}$$

Using the divisor equation [2,72], we find

$$\alpha \bullet_\tau \beta = \sum_{d \in \mathrm{Eff}_{\mathcal{X}}} \sum_{l \geqslant 0} \sum_{k=1}^{N} \frac{1}{l!} \langle \alpha, \beta, \tau', \dots, \tau', \phi_k \rangle^{\mathcal{X}}_{0,l+3,d} e^{\langle \tau_{0,2}, d\rangle} Q^d \phi^k. \tag{4}$$

Therefore, the quantum product can be viewed as a formal power series in $e^{\tau_{0,2}} Q$ and τ'. When this is a convergent power series, we can put $Q = 1$ and define

$$\circ_\tau := \bullet_\tau |_{Q=1}.$$

Under the following convergence assumption, the product $\circ_\tau$ defines an analytic family of commutative rings $(H_{\mathrm{orb}}(\mathcal{X}), \circ_\tau)$ over U:

Assumption 2.1. The orbifold quantum product $\circ_\tau$ is convergent over an open set $U \subset H^*_{\mathrm{orb}}(\mathcal{X})$ of the form:

$$U = \left\{ \tau \in H^*_{\mathrm{orb}}(\mathcal{X});\ \Re\langle \tau_{0,2}, d\rangle \leqslant -M,\ \forall d \in \mathrm{Eff}_{\mathcal{X}} \setminus \{0\},\ \|\tau'\| \leqslant e^{-M} \right\}$$

for a sufficiently big $M > 0$, where $\tau = \tau_{0,2} + \tau'$ is the decomposition in (3) and $\|\cdot\|$ is some norm on $H^*_{\mathrm{orb}}(\mathcal{X})$.

The domain U here contains the following limit direction:

$$\Re\langle \tau_{0,2}, d\rangle \to -\infty, \quad \forall d \in \mathrm{Eff}_{\mathcal{X}} \setminus \{0\},\ \tau' \to 0. \tag{5}$$

This is called the *large radius limit*. In the large radius limit, $\circ_\tau$ goes to the orbifold cup product $\cup_{\mathrm{orb}}$ due to Chen and Ruan [19]. (For manifolds, $\cup_{\mathrm{orb}}$ is the same as the cup product.)

2.2. Quantum D-modules and Galois action

We associate a meromorphic flat connection (quantum D-module) to the orbifold quantum cohomology. We introduce certain automorphisms of the quantum D-module, which we call *Galois actions*.

Take a homogeneous basis $\{\phi_k\}_{k=1}^N$ of $H^*_{\mathrm{orb}}(\mathcal{X})$ and let $\{t^k\}_{k=1}^N$ be the linear co-ordinate system on $H^*_{\mathrm{orb}}(\mathcal{X})$ dual to $\{\phi_k\}_{k=1}^N$. Let $\tau = \sum_{k=1}^N t^k \phi_k$ be a general point on $U \subset H^*_{\mathrm{orb}}(\mathcal{X})$. Let (τ, z) be a general point on $U \times \mathbb{C}$ and $(-) : U \times \mathbb{C} \to U \times \mathbb{C}$ be the map sending (τ, z) to $(\tau, -z)$.

Definition 2.2. The *quantum D-module $QDM(\mathcal{X})$* or *A-model D-module* is the tuple $(F, \nabla, (\cdot,\cdot)_F)$ consisting of the trivial holomorphic vector bundle $F := H^*(\mathcal{X}) \times (U \times \mathbb{C}) \to U \times \mathbb{C}$, the meromorphic flat connection ∇

$$\nabla_k = \nabla_{\frac{\partial}{\partial t^k}} = \frac{\partial}{\partial t^k} + \frac{1}{z}\phi_k \circ_\tau,$$
$$\nabla_{z\partial_z} = z\frac{\partial}{\partial z} - \frac{1}{z}E \circ_\tau + \mu,$$

and the ∇-flat pairing $(\cdot,\cdot)_F$:

$$(\cdot,\cdot)_F : (-)^*\mathcal{O}(F) \otimes \mathcal{O}(F) \to \mathcal{O}_{U\times\mathbb{C}}$$

induced from the orbifold Poincaré pairing $F_{(\tau,-z)} \times F_{(\tau,z)} = H^*_{\mathrm{orb}}(\mathcal{X}) \times H^*_{\mathrm{orb}}(\mathcal{X}) \to \mathbb{C}$. Here $E \in \mathcal{O}(F)$ is the *Euler vector field*

$$E := c_1(T\mathcal{X}) + \sum_{k=1}^{N}\left(1 - \frac{1}{2}\deg\phi_k\right)t^k\phi_k \tag{6}$$

and $\mu \in \mathrm{End}(H^*_{\mathrm{orb}}(\mathcal{X}))$ is the *Hodge grading operator*

$$\mu(\phi_k) := \left(\frac{1}{2}\deg\phi_k - \frac{n}{2}\right)\phi_k. \tag{7}$$

The flat connection ∇ is called the *Dubrovin connection* or *the first structure connection*. The standard argument (as in [28,57]) and the WDVV equation in orbifold Gromov–Witten theory [2] show that the Dubrovin connection is flat.

Note that the connection ∇ defines a map:

$$\nabla : \mathcal{O}(F) \longrightarrow \mathcal{O}(F)\big(U \times \{0\}\big) \otimes_{\mathcal{O}_{U\times\mathbb{C}}} \left(\pi^*\Omega^1_U \oplus \mathcal{O}_{U\times\mathbb{C}}\frac{dz}{z}\right),$$

where $\pi : U \times \mathbb{C} \to U$ is the projection. By identifying ϕ_i with the vector field $\partial/\partial t^i$, one can regard E as the vector field over U:

$$E = \sum_{k=1}^{N} r_k \frac{\partial}{\partial t^k} + \sum_{k=1}^{N}\left(1 - \frac{1}{2}\deg\phi_k\right)t^k\frac{\partial}{\partial t^k}, \tag{8}$$

where we put $c_1(\mathcal{X}) = \sum_{k=1}^{N} r_k\phi_k$. The Euler vector field satisfies the property:

$$\mathsf{Gr} := 2\left(\nabla_{z\partial_z} + \nabla_E + \frac{n}{2}\right) \quad \text{is regular at } z = 0. \tag{9}$$

The operator $\mathsf{Gr} : \mathcal{O}(F) \to \mathcal{O}(F)$ defines the grading for sections of F.

Let $H^2(\mathcal{X},\mathbb{Z})$ denote the cohomology of the constant sheaf $\mathbb{Z}$ on the topological *stack* $\mathcal{X}$ (not on the topological *space*). This group is the set of isomorphism classes of topological orbifold line bundles on $\mathcal{X}$. Let $L_\xi \to \mathcal{X}$ be the orbifold line bundle corresponding to $\xi \in H^2(\mathcal{X},\mathbb{Z})$. Let $0 \leqslant f_v(\xi) < 1$ be the rational number such that the stabilizer of $\mathcal{X}_v$ ($v \in \mathsf{T}$) acts on $L_\xi|_{\mathcal{X}_v}$ by a complex number $\exp(2\pi\mathbf{i}f_v(\xi))$. This number $f_v(\xi)$ is called the *age* of L_ξ along $\mathcal{X}_v$.

Proposition 2.3. *For $\xi \in H^2(\mathcal{X}, \mathbb{Z})$, the bundle isomorphism of F defined by*

$$\begin{aligned} H^*_{\mathrm{orb}}(\mathcal{X}) \times (U \times \mathbb{C}) &\longrightarrow H^*_{\mathrm{orb}}(\mathcal{X}) \times (U \times \mathbb{C}), \\ (\alpha, \tau, z) &\longmapsto \big(dG(\xi)\alpha, G(\xi)\tau, z\big) \end{aligned}$$

gives an automorphism of the quantum D-module, i.e. preserves the flat connection ∇ and the pairing $(\cdot,\cdot)_F$. Here $G(\xi), dG(\xi) : H^*_{\mathrm{orb}}(\mathcal{X}) \to H^*_{\mathrm{orb}}(\mathcal{X})$ are defined by

$$\begin{aligned} G(\xi)\bigg(\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} \tau_v\bigg) &= (\tau_0 - 2\pi \mathbf{i}\xi_0) \oplus \bigoplus_{v \in \mathsf{T}'} e^{2\pi \mathbf{i} f_v(\xi)} \tau_v, \\ dG(\xi)\bigg(\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} \tau_v\bigg) &= \tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} e^{2\pi \mathbf{i} f_v(\xi)} \tau_v, \end{aligned} \tag{10}$$

where $\tau_v \in H^*(\mathcal{X}_v)$ and ξ_0 is the image of ξ in $H^2(\mathcal{X}, \mathbb{Q})$. We call this Galois action *of $H^2(\mathcal{X}, \mathbb{Z})$ on $QDM(\mathcal{X})$.*

Proof. For $\alpha_1, \dots, \alpha_l \in H^*_{\mathrm{orb}}(\mathcal{X})$, we claim that

$$\langle \alpha_1, \alpha_2, \dots, \alpha_l \rangle_{0,l,d} = e^{-2\pi \mathbf{i}\langle \xi_0, d\rangle} \big\langle dG(\xi)\alpha_1, dG(\xi)\alpha_2, \dots, dG(\xi)\alpha_l \big\rangle_{0,l,d}.$$

If there exists an orbifold stable map $f : (C, x_1, \dots, x_l) \to \mathcal{X}$ of degree d, we have an orbifold line bundle f^*L_ξ on C such that the monodromy at x_k equals $\exp(2\pi \mathbf{i} f_{v_k}(\xi))$ where $\mathrm{ev}_k(f) \in \mathcal{X}_{v_k}$. Then we must have

$$\deg f^* L_\xi - \sum_{k=1}^{l} f_{v_k} \in \mathbb{Z}, \quad \text{i.e.} \quad e^{-2\pi \mathbf{i}\langle \xi_0, d\rangle} \prod_{i=1}^{l} e^{2\pi \mathbf{i} f_{v_i}(\xi)} = 1.$$

The claim follows from this. The lemma follows from this claim and (4). □

Without loss of generality, we can assume that U is invariant under the Galois action. By the Galois action, the quantum D-module descends to the quotient $F/H^2(\mathcal{X}, \mathbb{Z}) \to (U/H^2(\mathcal{X}, \mathbb{Z})) \times \mathbb{C}$. We refer to this flat connection over $(U/H^2(\mathcal{X}, \mathbb{Z})) \times \mathbb{C}$ also as the *quantum D-module*.

2.3. The space of solutions to the quantum differential equation

The equation $\nabla s = 0$ for a section s of F is called the *quantum differential equation*. A fundamental solution $L(\tau, z)$ to the quantum differential equation can be given by gravitational descendants. Let $\mathrm{pr} : I\mathcal{X} \to \mathcal{X}$ be the natural projection. We define the action of a class $\tau_0 \in H^*(\mathcal{X})$ on $H^*_{\mathrm{orb}}(\mathcal{X})$ by

$$\tau_0 \cdot \alpha = \mathrm{pr}^*(\tau_0) \cup \alpha, \quad \alpha \in H^*_{\mathrm{orb}}(\mathcal{X}),$$

where the right-hand side is the cup product on $I\mathcal{X}$. We define

$$L(\tau,z)\alpha := e^{-\tau_{0,2}/z}\alpha - \sum_{\substack{(d,l)\neq(0,0)\\ d\in\mathrm{Eff}_{\mathcal{X}},1\leqslant k\leqslant N}} \frac{\phi^k}{l!}\left\langle \phi_k,\tau',\ldots,\tau',\frac{e^{-\tau_{0,2}/z}\alpha}{z+\psi}\right\rangle_{0,l+2,d}^{\mathcal{X}} e^{\langle\tau_{0,2},d\rangle}, \quad (11)$$

where $\tau=\tau_{0,2}+\tau'$ is the decomposition in (3) and $1/(z+\psi)$ in the correlator should be expanded in the series $\sum_{k=0}^{\infty}(-1)^k z^{-k-1}\psi^k$. The following proposition is well-known for manifolds [28,63].

Proposition 2.4. *$L(\tau,z)$ satisfies the following differential equations:*

$$\nabla_k L(\tau,z)\alpha = 0, \qquad \nabla_{z\partial_z} L(\tau,z)\alpha = L(\tau,z)\left(\mu\alpha - \frac{\rho}{z}\alpha\right), \quad (12)$$

where $\alpha\in H^*_{\mathrm{orb}}(\mathcal{X})$, $\rho := c_1(T\mathcal{X})\in H^2(\mathcal{X})$ and μ is the grading operator (7)*. The flat section $L(\tau,z)\alpha$ (flat in the τ-direction) is characterized by the asymptotic initial condition:*

$$L(\tau,z)\alpha \sim e^{-\tau_{0,2}/z}\alpha \quad (13)$$

in the large radius limit (5) *with $\tau'=0$. Set*

$$z^{-\mu}z^{\rho} := \exp(-\mu\log z)\exp(\rho\log z).$$

Then we have

$$\nabla_k\big(L(\tau,z)z^{-\mu}z^{\rho}\alpha\big)=0, \quad \nabla_{z\partial_z}\big(L(\tau,z)z^{-\mu}z^{\rho}\alpha\big)=0, \quad (14)$$

$$\big(L(\tau,-z)\alpha, L(\tau,z)\beta\big)_{\mathrm{orb}} = (\alpha,\beta)_{\mathrm{orb}}, \quad (15)$$

$$dG(\xi)L\big(G(\xi)^{-1}\tau,z\big)\alpha = L(\tau,z)e^{-2\pi\mathbf{i}\xi_0/z}e^{2\pi\mathbf{i}f_v(\xi)}\alpha, \quad \text{if } \alpha\in H^*(\mathcal{X}_v), \quad (16)$$

where $dG(\xi)$, $G(\xi)$ are the Galois actions for $\xi\in H^2(\mathcal{X},\mathbb{Z})$ in Section 2.2.

Proof. The first equation of (12) follows from the topological recursion relation [72, 2.5.5] in orbifold Gromov–Witten theory. The proof for the case of manifolds can be found in [63, Proposition 2], [28, Chapter 10] and the proof for orbifolds is completely parallel.

For the second equation of (12), note that we can decompose L as $L(\tau,z)=S(\tau,z)\circ e^{-\tau_{0,2}/z}$ for some $\mathrm{End}(H^*_{\mathrm{orb}}(\mathcal{X}))$-valued function $S(\tau,z)$. The homogeneity of Gromov–Witten invariants shows that S preserves the degree, i.e. $(z\partial_z+E+\mu)\circ S(\tau,z)=S(\tau,z)\circ(z\partial_z+E+\mu)$, where E is regarded as the vector field (8). Therefore, $(z\partial_z+E+\mu)\circ L(\tau,z)=L(\tau,z)\circ(z\partial_z+E+\mu-\rho/z)$. The second equation of (12) follows from this and the first equation.

The asymptotic initial condition (13) is obvious from the definition (11).

Eq. (14) follows from (12) and the fact that $z^{-\mu}z^{\rho}\alpha$ satisfies the differential equation $(z\partial_z+\mu-\rho/z)(z^{-\mu}z^{\rho}\alpha)=0$, which follows easily from the commutation relation $[\mu,\rho]=\rho$.

To show Eq. (15), put $\boldsymbol{s}' = L(\tau, -z)\alpha$ and $\boldsymbol{s} = L(\tau, z)\beta$. By using (12) and the Frobenius property $(\alpha \circ_\tau \beta, \gamma)_{\mathrm{orb}} = (\alpha, \beta \circ_\tau \gamma)_{\mathrm{orb}}$, we have

$$\frac{\partial}{\partial t^k}(\boldsymbol{s}', \boldsymbol{s})_{\mathrm{orb}} = \frac{1}{z}(\phi_k \circ_\tau \boldsymbol{s}', \boldsymbol{s})_{\mathrm{orb}} - \frac{1}{z}(\boldsymbol{s}', \phi_k \circ_\tau \boldsymbol{s})_{\mathrm{orb}} = 0.$$

Hence $(\boldsymbol{s}', \boldsymbol{s})_{\mathrm{orb}}$ is constant in τ. Using the asymptotics $\boldsymbol{s}' \sim e^{\tau_{0,2}/z}\alpha$ and $\boldsymbol{s} \sim e^{-\tau_{0,2}/z}\beta$, we have

$$(\boldsymbol{s}', \boldsymbol{s})_{\mathrm{orb}} \sim \left(e^{-\tau_{0,2}/z}\alpha, e^{\tau_{0,2}/z}\beta\right)_{\mathrm{orb}} = (\alpha, \beta)_{\mathrm{orb}}$$

and Eq. (15) follows.

Since the Galois action preserves ∇, it follows that $dG(\xi)L(G(\xi)^{-1}\tau, z)\alpha$ is flat in the τ-direction. Eq. (16) follows from the characterization (13) and the asymptotics

$$dG(\xi)L\left(G(\xi)^{-1}\tau, z\right)\alpha \sim e^{-\tau_{0,2}/z}e^{-2\pi \mathbf{i}\xi_0/z}e^{2\pi\mathbf{i} f_v(\xi)}\alpha. \qquad \Box$$

Although the convergence of $L(\tau, z)$ is not a priori clear, we know from the differential equations above and the convergence assumption of $\circ_\tau$ that $L(\tau, z)$ is convergent on $(\tau, z) \in U \times \mathbb{C}^*$.

Definition 2.5. The space $\mathcal{S}(\mathcal{X})$ of multi-valued ∇-flat sections of the quantum D-module $(F, \nabla, (\cdot,\cdot)_F)$ is defined to be

$$\mathcal{S}(\mathcal{X}) := \left\{s \in \Gamma\left(U \times \widetilde{\mathbb{C}^*}, \mathcal{O}(F)\right);\ \nabla s = 0\right\},$$

where $\widetilde{\mathbb{C}^*}$ is the universal cover of $\mathbb{C}^*$. This is a finite-dimensional $\mathbb{C}$-vector space with $\dim \mathcal{S}(\mathcal{X}) = \dim H^*_{\mathrm{orb}}(\mathcal{X})$. The pairing $(\cdot,\cdot)_{\mathcal{S}}$ on $\mathcal{S}(\mathcal{X})$ is given by

$$(s_1, s_2)_{\mathcal{S}} := \left(s_1\left(\tau, e^{\pi\mathbf{i}}z\right), s_2(\tau, z)\right)_{\mathrm{orb}} \in \mathbb{C}, \tag{17}$$

where $s_1(\tau, e^{\pi\mathbf{i}}z)$ is the parallel translate of $s_1(\tau, z)$ along the counter-clockwise path $[0,1] \ni \theta \mapsto e^{\mathbf{i}\pi\theta}z$. Note that the right-hand side is a complex number which does not depend on (τ, z). The Galois action in Proposition 2.3 defines an automorphism of $\mathcal{S}(\mathcal{X})$ for $\xi \in H^2(\mathcal{X}, \mathbb{Z})$:

$$G^{\mathcal{S}}(\xi) : \mathcal{S}(\mathcal{X}) \to \mathcal{S}(\mathcal{X}), \qquad s(\tau, z) \mapsto dG(\xi)s\left(G(\xi)^{-1}\tau, z\right). \tag{18}$$

Using the fundamental solution in Proposition 2.4, we define the *cohomology framing* $\mathcal{Z}_{\mathrm{coh}} : H^*_{\mathrm{orb}}(\mathcal{X}) \to \mathcal{S}(\mathcal{X})$ of $\mathcal{S}(\mathcal{X})$ by

$$\mathcal{Z}_{\mathrm{coh}}(\alpha) := L(\tau, z)z^{-\mu}z^{\rho}\alpha. \tag{19}$$

The pairing and the Galois action on $\mathcal{S}(\mathcal{X})$ can be written in terms of the cohomology framing as

$$\begin{aligned} \left(\mathcal{Z}_{\mathrm{coh}}(\alpha), \mathcal{Z}_{\mathrm{coh}}(\beta)\right)_{\mathcal{S}} &= \left(e^{\pi\mathbf{i}\rho}\alpha, e^{\pi\mathbf{i}\mu}\beta\right)_{\mathrm{orb}}, \\ G^{\mathcal{S}}(\xi)\left(\mathcal{Z}_{\mathrm{coh}}(\alpha)\right) &= \mathcal{Z}_{\mathrm{coh}}\left(\left(\bigoplus_{v\in\mathsf{T}} e^{-2\pi\mathbf{i}\xi_0}e^{2\pi\mathbf{i} f_v(\xi)}\right)\alpha\right). \end{aligned} \tag{20}$$

Here $\xi_0 \in H^2(\mathcal{X}, \mathbb{Q})$ and $f_v(\xi) \in [0,1) \cap \mathbb{Q}$ are introduced before Proposition 2.3. The first equation follows from (15) and the second equation follows from (16).

The Galois actions on $\mathcal{S}(\mathcal{X})$ can be viewed as the monodromy transformations of the flat bundle $F/H^2(\mathcal{X},\mathbb{Z}) \to (U/H^2(\mathcal{X},\mathbb{Z})) \times \mathbb{C}^*$ in the τ-direction. The monodromy with respect to z is given by

$$\left[\mathcal{Z}_{\mathrm{coh}}(\alpha)\right]_{z\mapsto e^{2\pi\mathbf{i}}z} = \mathcal{Z}_{\mathrm{coh}}\left(e^{-2\pi\mathbf{i}\mu}e^{2\pi\mathbf{i}\rho}\alpha\right). \tag{21}$$

This coincides with the Galois action $(-1)^n G^{\mathcal{S}}([K_{\mathcal{X}}])$ and also corresponds to the Serre functor of the derived category $D(\mathcal{X})$. Here, $[K_{\mathcal{X}}]$ is the class of the canonical line bundle. When $\mathcal{X}$ is Calabi–Yau, i.e. $K_{\mathcal{X}}$ is trivial, the pairing $(\cdot,\cdot)_{\mathcal{S}}$ is either symmetric or anti-symmetric depending on whether n is even or odd. In general, this pairing is neither symmetric nor anti-symmetric.

2.4. $\widehat{\Gamma}$-integral structure

By an *integral structure* in quantum cohomology we mean a $\mathbb{Z}$-local system $F_{\mathbb{Z}} \to U \times \mathbb{C}^*$ underlying the flat bundle $(F,\nabla)|_{U\times\mathbb{C}^*}$. This is given by an integral lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ in the space $\mathcal{S}(\mathcal{X})$ of multi-valued flat sections of $QDM(\mathcal{X})$. There are a priori many choices of integral lattices in $\mathcal{S}(\mathcal{X})$. We introduce the $\widehat{\Gamma}$-integral structure which has several nice properties.

Let $K(\mathcal{X})$ denote the Grothendieck group of topological orbifold vector bundles on $\mathcal{X}$. See e.g. [3,58] for vector bundles on orbifolds. For an orbifold vector bundle $\widetilde{V}$ on the inertia stack $I\mathcal{X}$, we have an eigenbundle decomposition of $\widetilde{V}|_{\mathcal{X}_v}$

$$\widetilde{V}|_{\mathcal{X}_v} = \bigoplus_{0\leqslant f<1} \widetilde{V}_{v,f}$$

with respect to the action of the stabilizer of $\mathcal{X}_v$. Here, the stabilizer acts on $\widetilde{V}_{v,f}$ by $\exp(2\pi\mathbf{i}f) \in \mathbb{C}$. Let $\mathrm{pr}: I\mathcal{X} \to \mathcal{X}$ be the projection. The Chern character $\widetilde{\mathrm{ch}}: K(\mathcal{X}) \to H^*(I\mathcal{X})$ is defined for an orbifold vector bundle V on $\mathcal{X}$ by

$$\widetilde{\mathrm{ch}}(V) := \bigoplus_{v\in\mathsf{T}} \sum_{0\leqslant f<1} e^{2\pi\mathbf{i}f}\,\mathrm{ch}\left((\mathrm{pr}^* V)_{v,f}\right)$$

where ch is the ordinary Chern character. For an orbifold vector bundle V on $\mathcal{X}$, let $\delta_{v,f,i}$, $i = 1,\ldots,l_{v,f}$, be the Chern roots of $(\mathrm{pr}^* V)_{v,f}$. The Todd class $\widetilde{\mathrm{Td}}: K(\mathcal{X}) \to H^*(I\mathcal{X})$ is defined by

$$\widetilde{\mathrm{Td}}(V) = \bigoplus_{v\in\mathsf{T}} \prod_{0<f<1,1\leqslant i\leqslant l_{v,f}} \frac{1}{1-e^{-2\pi\mathbf{i}f}e^{-\delta_{v,f,i}}} \prod_{f=0,1\leqslant i\leqslant l_{v,0}} \frac{\delta_{v,0,i}}{1-e^{-\delta_{v,0,i}}}.$$

These characteristic classes appear in the following theorem.

Theorem 2.6 *(Orbifold Riemann–Roch). (See [54,70].) Assume that $\mathcal{X}$ has the resolution property (see e.g.* [71]*). For a holomorphic orbifold vector bundle V on $\mathcal{X}$, the Euler characteristic $\chi(V)$ is given by*

$$\chi(V) := \sum_{i=0}^{\dim\mathcal{X}} (-1)^i \dim H^i(\mathcal{X},V) = \int_{I\mathcal{X}} \widetilde{\mathrm{ch}}(V)\cup\widetilde{\mathrm{Td}}(T\mathcal{X}). \tag{22}$$

Define a multiplicative characteristic class $\widehat{\Gamma}: K(\mathcal{X}) \to H^*(I\mathcal{X})$ by

$$\widehat{\Gamma}(V) := \bigoplus_{v \in \mathsf{T}} \prod_{0 \leqslant f < 1} \prod_{i=1}^{l_{v,f}} \Gamma(1 - f + \delta_{v,f,i}) \in H^*(I\mathcal{X}), \tag{23}$$

where $\delta_{v,f,i}$ is the same as above. The Gamma function on the right-hand side should be expanded in series at $1 - f > 0$. We assume the following conditions.

Assumption 2.7.

(a) The map $\widetilde{\mathrm{ch}}: K(\mathcal{X}) \to H^*(I\mathcal{X})$ becomes an isomorphism after tensored with $\mathbb{C}$.
(b) The right-hand side of the orbifold Riemann–Roch formula (22) takes values in $\mathbb{Z}$ for any (not necessarily holomorphic) complex orbifold vector bundle V on $\mathcal{X}$. Define $\chi(V)$ to be the value of the right-hand side of (22) for any orbifold vector bundle V.
(c) The pairing $(V_1, V_2) \mapsto \chi(V_1 \otimes V_2)$ on $K(\mathcal{X})$ induces a surjective map $K(\mathcal{X}) \to \mathrm{Hom}(K(\mathcal{X}), \mathbb{Z})$.

Remark 2.8.

(i) When $\mathcal{X}$ can be presented as a quotient $[Y/G]$ as a topological orbifold, where Y is a compact manifold and G is a compact Lie group acting on Y with at most finite stabilizers, Part (a) of the assumption follows from Adem–Ruan's decomposition theorem [3, Theorem 5.1]. Note that an orbifold without generic stabilizers can be presented as a quotient orbifold $[Y/G]$ (see e.g. [3]).
(ii) When $\mathcal{X}$ is again a quotient orbifold $[Y/G]$, Part (b) follows from Kawasaki's index theorem [55] for elliptic operators on orbifolds (whose proof uses the G-equivariant index). The right-hand side of (22) becomes the index of a certain elliptic operator $\bar{\partial} + \bar{\partial}^*: V \otimes \Omega_{\mathcal{X}}^{0,\mathrm{even}} \to V \otimes \Omega_{\mathcal{X}}^{0,\mathrm{odd}}$, where $\bar{\partial}$ is a not necessarily integrable $(0,1)$ connection and $\bar{\partial}^*$ is its adjoint. The author does not know a purely topological proof.
(iii) Part (c) would follow from a universal coefficient theorem and Poincaré duality for orbifold K-theory (which are true for manifolds), but the author does not know a proof nor a reference.

Definition 2.9. We define the *K-group framing* $\mathcal{Z}_K : K(\mathcal{X}) \to \mathcal{S}(\mathcal{X})$ of the space $\mathcal{S}(\mathcal{X})$ of multi-valued flat sections of the quantum D-module by the formula:

$$\begin{aligned} &\mathcal{Z}_K(V) := \mathcal{Z}_{\mathrm{coh}}\big(\Psi(V)\big) = L(\tau, z) z^{-\mu} z^{\rho} \Psi(V), \\ &\quad \text{where } \Psi(V) := (2\pi)^{-n/2} \widehat{\Gamma}(T\mathcal{X}) \cup (2\pi \mathbf{i})^{\deg/2} \mathrm{inv}^*\big(\widetilde{\mathrm{ch}}(V)\big). \end{aligned} \tag{24}$$

Here $\deg: H^*(I\mathcal{X}) \to H^*(I\mathcal{X})$ is a grading operator on $H^*(I\mathcal{X})$ defined by $\deg = 2k$ on $H^{2k}(I\mathcal{X})$[2] and $\cup$ is the cup product in $H^*(I\mathcal{X})$. We call the image $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} := \mathcal{Z}_K(K(\mathcal{X}))$ of the K-group framing the *$\widehat{\Gamma}$-integral structure*.

[2] Note that deg is the degree of the cohomology class as an element of $H^*(I\mathcal{X})$, *not* as an element of $H^*_{\mathrm{orb}}(\mathcal{X})$.

Proposition 2.10. *Assume Assumption* 2.7. *The $\widehat{\Gamma}$-integral structure $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ satisfies the following properties.*

(i) *By Part* (a) *of the assumption, $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ is a $\mathbb{Z}$-lattice in $\mathcal{S}(\mathcal{X})$:*

$$\mathcal{S}(\mathcal{X}) = \mathcal{S}(\mathcal{X})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}.$$

(ii) *The Galois action $G^{\mathcal{S}}(\xi)$ on $\mathcal{S}(\mathcal{X})$ in* (18) *preserves the lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ and corresponds to the tensor by the line bundle $L_\xi^\vee$ in $K(\mathcal{X})$:*

$$\mathcal{Z}_K\big(L_\xi^\vee \otimes V\big) = G^{\mathcal{S}}(\xi)\big(\mathcal{Z}_K(V)\big)$$

where L_ξ is the line bundle corresponding to $\xi \in H^2(\mathcal{X}, \mathbb{Z})$.

(iii) *The pairing $(\cdot,\cdot)_{\mathcal{S}}$ on $\mathcal{S}(\mathcal{X})$ in* (17) *corresponds to the Mukai pairing on $K(\mathcal{X})$ defined by $(V_1, V_2)_{K(\mathcal{X})} := \chi(V_2^\vee \otimes V_1)$:*

$$\big(\mathcal{Z}_K(V_1), \mathcal{Z}_K(V_2)\big)_{\mathcal{S}} = (V_1, V_2)_{K(\mathcal{X})}.$$

In particular, the pairing $(\cdot,\cdot)_{\mathcal{S}(\mathcal{X})}$ restricted on $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ takes values in $\mathbb{Z}$ by Part (b) *of the assumption and is unimodular by Part* (c).

Proof. Because $\widehat{\Gamma}_{\mathcal{X}}\cup$ and $(2\pi\mathbf{i})^{\deg/2}$ are invertible operators over $\mathbb{C}$, Part (a) of Assumption 2.7 implies (i). It is easy to check the second statement (ii). For (iii), we calculate

$$\begin{aligned}
\big(\mathcal{Z}_K(V_1), \mathcal{Z}_K(V_2)\big)_{\mathcal{S}} &= \big(e^{\pi\mathbf{i}\rho}\Psi(V_1), e^{\pi\mathbf{i}\mu}\Psi(V_2)\big)_{\mathrm{orb}} \quad \text{by (20)} \\
&= \frac{1}{(2\pi)^n}\sum_{v\in\mathsf{T}}\int_{\mathcal{X}_v}\big(e^{\pi\mathbf{i}\rho}\widehat{\Gamma}(T\mathcal{X})_{\mathrm{inv}(v)}(2\pi\mathbf{i})^{\frac{\deg}{2}}\widetilde{\mathrm{ch}}(V_1)_v\big) \\
&\quad \cup \big(e^{\pi\mathbf{i}(\iota_v-\frac{n}{2}+\frac{\deg}{2})}\widehat{\Gamma}(T\mathcal{X})_v(2\pi\mathbf{i})^{\frac{\deg}{2}}\widetilde{\mathrm{ch}}(V_2)_{\mathrm{inv}(v)}\big) \quad \text{by (24)} \\
&= \frac{1}{(2\pi)^n}\sum_{v\in\mathsf{T}}(2\pi\mathbf{i})^{\dim\mathcal{X}_v}\int_{\mathcal{X}_v}\prod_{f,i}\Gamma\left(1-\overline{f}+\frac{\delta_{v,f,i}}{2\pi\mathbf{i}}\right)\Gamma\left(1-f-\frac{\delta_{v,f,i}}{2\pi\mathbf{i}}\right) \\
&\quad \cdot e^{\frac{\rho}{2}}\widetilde{\mathrm{ch}}(V_1)_v\cdot e^{\pi\mathbf{i}(\iota_v-\frac{n}{2}+\frac{\deg}{2})}\widetilde{\mathrm{ch}}(V_2)_{\mathrm{inv}(v)},
\end{aligned}$$

where α_v denotes the v-component of $\alpha \in H^*_{\mathrm{orb}}(\mathcal{X})$. We used the fact that $\mu|_{H^*(\mathcal{X}_v)} = \iota_v - \frac{n}{2} + \frac{\deg}{2}$ in the second step and that $\int_{\mathcal{X}_v}((2\pi\mathbf{i})^{\frac{\deg}{2}}\alpha) = (2\pi\mathbf{i})^{\dim\mathcal{X}_v}\int_{\mathcal{X}_v}\alpha$ in the third step. We also used the fact that $\{\delta_{\mathrm{inv}(v),f,i}\}_i = \{\delta_{v,\overline{f},i}\}_i$, where

$$\overline{f} := \begin{cases} 1-f & \text{if } 0<f<1, \\ 0 & \text{if } f=0. \end{cases}$$

Using $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$ and $\sum_{f,i} \delta_{v,f,i} = \mathrm{pr}^* \rho|_{\mathcal{X}_v}$, we calculate

$$\prod_{f,i} \Gamma\left(1 - \overline{f} + \frac{\delta_{v,f,i}}{2\pi\mathbf{i}}\right)\Gamma\left(1 - f - \frac{\delta_{v,f,i}}{2\pi\mathbf{i}}\right) = (2\pi\mathbf{i})^{n-\dim \mathcal{X}_v} e^{-\frac{\rho}{2}} e^{-\pi \mathbf{i}\iota_v} \widetilde{\mathrm{Td}}(T\mathcal{X})_v.$$

The conclusion follows from the orbifold Riemann–Roch (22). □

The lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} \subset \mathcal{S}(\mathcal{X})$ defines a $\mathbb{Z}$-local system $F_{\mathbb{Z}} \to U \times \mathbb{C}^*$ underlying the flat vector bundle $(F|_{U\times\mathbb{C}^*}, \nabla)$. Because $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ is invariant under the Galois action, the local system $F_{\mathbb{Z}} \to U \times \mathbb{C}^*$ descends to a local system over $(U/H^2(\mathcal{X},\mathbb{Z})) \times \mathbb{C}^*$.

Remark 2.11. When we consider the *algebraic part* of quantum cohomology, we can instead use the K-group of algebraic vector bundles or coherent sheaves to define an integral structure. Let $A^*(\mathcal{X})_{\mathbb{C}}$ denote the Chow ring of $\mathcal{X}$ over $\mathbb{C}$. We set $\mathbb{H}^*(\mathcal{X}_v) := \mathrm{Im}(A^*(\mathcal{X}_v)_{\mathbb{C}} \to H^*(\mathcal{X}_v))$ and define $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X}) := \bigoplus_{v\in\mathsf{T}} \mathbb{H}^*(\mathcal{X}_v)$. Under Assumption 2.1, the algebraic quantum D-module is defined to be the holomorphic vector bundle

$$\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X}) \times (U' \times \mathbb{C}) \to (U' \times \mathbb{C}), \quad U' = U \cap \mathbb{H}^*_{\mathrm{orb}}(\mathcal{X})$$

endowed with the restriction of the Dubrovin connection to U' and the orbifold Poincaré pairing. The Galois action on it is given by an element of $\mathrm{Pic}(\mathcal{X})$. Here we used the fact that the quantum product among classes in $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X})$ again belongs to $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X})$; this follows from the algebraic construction of orbifold Gromov–Witten theory [2]. When we assume Hodge conjecture for all $\mathcal{X}_v$, each $\mathbb{H}^*(\mathcal{X}_v)$ has Poincaré duality and the orbifold Poincaré pairing is non-degenerate on $\mathbb{H}^*_{\mathrm{orb}}(\mathcal{X})$. Definition 2.9 applies to this algebraic quantum D-module with $K(\mathcal{X})$ being the algebraic K-group.

We introduce the *quantum cohomology central charge* of $V \in K(\mathcal{X})$ associated to the $\widehat{\Gamma}$-class to be the function:

$$Z(V)(\tau,z) := c(z)\int_{\mathcal{X}} \mathcal{Z}_K(V)(\tau,z) = c(z)\big(\mathbf{1}, \mathcal{Z}_K(V)(\tau,z)\big)_{\mathrm{orb}} \tag{25}$$

where $c(z) = (2\pi z)^{n/2}/(2\pi\mathbf{i})^n$ is a normalization factor, cf. Hosono's central charge formula [45, Definition 2.1] for a Calabi–Yau $\mathcal{X}$ given in terms of periods of the mirror. For Calabi–Yau 3-folds, the author hopes that our $Z(V)$ gives the physics central charge of the B-type D-brane in the class V. This plays an important role in the Douglas–Bridgeland stability on derived categories [13,31].

2.5. Givental's symplectic space, $\frac{\infty}{2}$VHS and J-function

Givental's symplectic space [22,39] is the loop space on $H^*_{\mathrm{orb}}(\mathcal{X})$ with a loop parameter z. This is identified with the space of sections of $QDM(\mathcal{X})$ which are flat only in the τ-direction. In the Givental space, $QDM(\mathcal{X})$ can be realized as moving semi-infinite subspaces. This is an example of *semi-infinite variation of Hodge structure* ($\frac{\infty}{2}$VHS for short) due to Barannikov [7,8]. The J-function is the image of the unit section $\mathbf{1}$ in this realization. The notion of $\frac{\infty}{2}$VHS will be used only in Section 5.

Definition 2.12. Let $\mathcal{O}(\mathbb{C}^*)$ denote the space of holomorphic functions on $\mathbb{C}^*$ with the coordinate z. The *Givental space* $\mathcal{H}$ is defined to be the free $\mathcal{O}(\mathbb{C}^*)$-module:

$$\mathcal{H} = H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C}^*) \tag{26}$$

endowed with the pairing $(\cdot,\cdot)_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathcal{O}(\mathbb{C}^*)$

$$\big(\alpha(z), \beta(z)\big)_{\mathcal{H}} := \big(\alpha(-z), \beta(z)\big)_{\mathrm{orb}} \tag{27}$$

and the symplectic form $\Omega(\alpha(z), \beta(z)) = \mathrm{Res}_{z=0}\, dz(\alpha(z), \beta(z))_{\mathcal{H}}$. Using the fundamental solution $L(\tau, z)$, we identify $\mathcal{H}$ with the space of sections of $QDM(\mathcal{X})$ which are flat in the τ-direction,

$$\mathcal{H} \ni \alpha \longmapsto L(\tau, z)\alpha \in \Gamma\big(U \times \mathbb{C}^*, \mathcal{O}(F)\big). \tag{28}$$

Note that under this identification, $(\cdot,\cdot)_{\mathcal{H}}$ corresponds to $(\cdot,\cdot)_F$ by (15). The Galois action on flat sections (18) induces a map $G^{\mathcal{H}}(\xi) : \mathcal{H} \to \mathcal{H}$:

$$G^{\mathcal{H}}(\xi)\bigg(\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} \tau_v\bigg) = e^{-2\pi\mathbf{i}\xi_0/z}\tau_0 \oplus \bigoplus_{v \in \mathsf{T}'} e^{-2\pi\mathbf{i}\xi_0/z} e^{2\pi\mathbf{i} f_v(\xi)}\tau_v, \tag{29}$$

by (16). Here we used the decomposition $\mathcal{H}^{\mathcal{X}} = \bigoplus_{v \in \mathsf{T}} H^*(\mathcal{X}_v) \otimes \mathcal{O}(\mathbb{C}^*)$.

We introduce the $\frac{\infty}{2}$VHS associated to quantum cohomology. Let $\pi : U \times \mathbb{C} \to U$ be the natural projection. Under the identification (28), the fiber $(\pi_*\mathcal{O}(F))_\tau$ at $\tau \in U$ is identified with the semi-infinite subspace $\mathbb{F}_\tau$ of $\mathcal{H}$:

$$\mathbb{F}_\tau := \mathbb{J}_\tau\big(H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C})\big), \quad \mathbb{J}_\tau := L(\tau, z)^{-1}.$$

We call $\mathbb{F}_\tau$ the *semi-infinite Hodge structure*. This satisfies the following properties:

- $X\mathbb{F}_\tau \subset z^{-1}\mathbb{F}_\tau$ for a tangent vector $X \in T_\tau U$;
- $\mathbb{F}_\tau$ is isotropic with respect to Ω, i.e. $(\mathbb{F}_\tau, \mathbb{F}_\tau)_{\mathcal{H}} \subset \mathcal{O}(\mathbb{C})$;
- $(2E + \nabla_{z\partial_z})\mathbb{F}_\tau \subset \mathbb{F}_\tau$.

Here we regard $\tau \mapsto \mathbb{F}_\tau$ as a holomorphic map from U to the Segal–Wilson Grassmannian (see e.g. [65]). Also $\nabla_{z\partial_z}$ denotes the operator on $\mathcal{H}$ induced from $\nabla_{z\partial_z}$. We call the family $\tau \mapsto \mathbb{F}_\tau$ *(a moving subspace realization of) the quantum cohomology* $\frac{\infty}{2}$*VHS*. The first property is an analogue of Griffiths transversality and the second is the Hodge–Riemann bilinear relation. We refer the reader to [27, Section 2], [49, Section 2] for the details.

Remark 2.13. The $\frac{\infty}{2}$VHS defines a Lagrangian cone $\mathcal{L}$ in $\mathcal{H}$:

$$\mathcal{L} := \bigcup_{\tau \in U} z\mathbb{F}_\tau. \tag{30}$$

This plays an important role in Givental's theory. The cone $\mathcal{L}$ can be written as the graph of the differential $d\mathcal{F}_0$ of the genus zero descendant potential $\mathcal{F}_0$ (with a dilaton shift). We refer the reader to [22] for this connection.

Using the fact that $L(\tau,z)^{-1}$ is the adjoint of $L(\tau,-z)$ with respect to the orbifold Poincaré pairing (see (15)), we can calculate the embedding $\mathbb{J}_\tau = L(\tau,z)^{-1} : (\pi_*\mathcal{O}(F))_\tau \hookrightarrow \mathcal{H}$ explicitly as follows:

$$\mathbb{J}_\tau \alpha = e^{\tau_{0,2}/z}\left(\alpha + \sum_{\substack{(d,l)\neq(0,0)\\ d\in \mathrm{Eff}_{\mathcal{X}}}} \sum_{i=1}^{N} \frac{1}{l!}\left\langle \alpha, \tau', \dots, \tau', \frac{\phi_i}{z-\psi}\right\rangle_{0,l+2,d}^{\mathcal{X}} e^{\langle \tau_{0,2}, d\rangle}\phi^i\right). \tag{31}$$

Definition 2.14. The *J-function* [22,28,38] is the image of the unit section $\mathbf{1}$ under the embedding $\mathbb{J}_\tau : (\pi_*\mathcal{O}(F))_\tau \hookrightarrow \mathcal{H}$: $J(\tau,z) := \mathbb{J}_\tau \mathbf{1} = L(\tau,z)^{-1}\mathbf{1}$. Because the unit section $\mathbf{1}$ is invariant under the Galois action, we have

$$J\big(G(\xi)\tau, z\big) = G^{\mathcal{H}}(\xi) J(\tau,z) \tag{32}$$

which follows from (16).

The J-function is the unit section $\mathbf{1}$ expressed in the τ-flat frame $L(\tau,z)$. The *H-function* $H_K(\tau,z)$ is defined to be the $K(\mathcal{X})\otimes\mathbb{C}$-valued function which expresses $\mathbf{1}$ in terms of the K-group framing (24):

$$H_K(\tau,z) := c\big(e^{-\pi \mathbf{i}}z\big)\cdot \Psi^{-1}\big(z^{-\rho}z^{\mu}L(\tau,z)^{-1}\mathbf{1}\big), \tag{33}$$

i.e. $c(e^{-\pi\mathbf{i}}z)\mathbf{1} = \mathcal{Z}_K(H_K(\tau,z))(\tau,z)$. Here $c(e^{-\pi\mathbf{i}}z) := (2\pi z)^{n/2}/(-2\pi)^n$ is a normalization factor. We also use $H^*(I\mathcal{X})$-valued function $H(\tau,z) := \widetilde{\mathrm{ch}}(H_K(\tau,z))$. The quantum cohomology central charge (25) can be written as (cf. [45, Eq. (2.3)]):

$$Z(V)(\tau,z) = \chi\big(H_K\big(\tau, e^{\pi\mathbf{i}}z\big)\otimes V^\vee\big) = \int_{I\mathcal{X}} H\big(\tau, e^{\pi\mathbf{i}}z\big)\cup \widetilde{\mathrm{ch}}\big(V^\vee\big)\cup \widetilde{\mathrm{Td}}(T\mathcal{X}). \tag{34}$$

Proof. We have $Z(V)(\tau,z) = (\mathcal{Z}_K(H_K(\tau,e^{\pi\mathbf{i}}z))(\tau,e^{\pi\mathbf{i}}z), \mathcal{Z}_K(V)(\tau,z))_{\mathrm{orb}}$. The formulas follows from this, Proposition 2.10, (iii) and orbifold Riemann–Roch (22). $\square$

3. Landau–Ginzburg mirror of toric orbifolds

In this section, we describe the Landau–Ginzburg (LG) models which are mirror to compact toric orbifolds. The LG mirrors for toric manifolds have been proposed by Givental [37,38] and Hori and Vafa [43] and they are easily adapted to the case of toric orbifolds. We also construct a meromorphic flat connection (B-model D-module) over the product of $\mathbb{C}$ with the parameter space $\mathcal{M}$ of the LG models. The B-model D-module has been studied in singularity theory as the Brieskorn lattice. We give an analytical construction based on oscillatory integrals. See Sabbah [67] for an algebraic construction (for a tame function on an algebraic variety) using the Fourier–Laplace transform of the algebraic Gauß–Manin system (see also [30,68]).

3.1. Toric orbifolds

To fix the notation, we give the definition of toric orbifolds and collect several facts. By a toric orbifold, we mean a toric Deligne–Mumford stack in the sense of Borisov, Chen, and Smith [12]. We only deal with a compact toric orbifold with a projective coarse moduli space and define a toric orbifold as a quotient of $\mathbb{C}^m$ by an algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^r$. The basic references for toric varieties (orbifolds) are made to [6,12,35,61].

3.1.1. Definition

We begin with the following data:

- an r-dimensional algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^r$; we set $\mathbb{L} := \mathrm{Hom}(\mathbb{C}^*, \mathbb{T})$;
- m elements $D_1, \dots, D_m \in \mathbb{L}^\vee = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ such that $\mathbb{L}^\vee \otimes \mathbb{R} = \sum_{i=1}^m \mathbb{R} D_i$;
- a vector $\eta \in \mathbb{L}^\vee \otimes \mathbb{R}$.

The elements $D_1, \dots, D_m$ define a homomorphism $\mathbb{T} \to (\mathbb{C}^*)^m$. Let $\mathbb{T}$ act on $\mathbb{C}^m$ via this homomorphism. The vector η defines a stability condition of this torus action. Set

$$\mathcal{A} := \left\{ I \subset \{1, \dots, m\};\ \sum_{i \in I} \mathbb{R}_{>0} D_i \ni \eta \right\}.$$

A toric orbifold $\mathcal{X}$ is defined to be the quotient stack

$$\mathcal{X} = [\mathcal{U}_\eta / \mathbb{T}], \quad \mathcal{U}_\eta := \mathbb{C}^m \Big\backslash \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I,$$

where $\mathbb{C}^I := \{(z_1, \dots, z_m) \in \mathbb{C}^m;\ z_i = 0 \text{ for } i \notin I\}$. Under the following conditions, $\mathcal{X}$ is a smooth Deligne–Mumford stack with a projective coarse moduli space:

(A) $\{1, \dots, m\} \in \mathcal{A}$.
(B) $\sum_{i \in I} \mathbb{R} D_i = \mathbb{L}^\vee \otimes \mathbb{R}$ for $I \in \mathcal{A}$.
(C) $\{(c_1, \dots, c_m) \in \mathbb{R}^m_{\geqslant 0};\ \sum_{i=1}^m c_i D_i = 0\} = \{0\}$.

The conditions (A), (B) and (C) ensure that $\mathcal{X}$ is non-empty, that the stabilizer is finite and that $\mathcal{X}$ is compact respectively. The generic stabilizer of $\mathcal{X}$ is given by the kernel of $\mathbb{T} \to (\mathbb{C}^*)^m$ and $\dim_{\mathbb{C}} \mathcal{X} = n := m - r$.

We can also construct $\mathcal{X}$ as a symplectic quotient as follows (see also [6]). Let $\mathbb{T}_{\mathbb{R}}$ denote the maximal compact subgroup of $\mathbb{T}$ isomorphic to $(S^1)^r$. Let $\mathfrak{h} : \mathbb{C}^m \to \mathbb{L}^\vee \otimes \mathbb{R}$ be the moment map for the $\mathbb{T}_{\mathbb{R}}$-action on $\mathbb{C}^m$:

$$\mathfrak{h}(z_1, \dots, z_m) = \sum_{i=1}^m |z_i|^2 D_i.$$

The $\mathbb{T}_{\mathbb{R}}$-action on the level set $\mathfrak{h}^{-1}(\eta)$ has only finite stabilizers and we have an isomorphism of symplectic orbifolds:

$$\mathcal{X} \cong \mathfrak{h}^{-1}(\eta) / \mathbb{T}_{\mathbb{R}}. \tag{35}$$

By renumbering the indices if necessary, we can assume that

$$\{1,\dots,m\}\setminus\{i\}\in\mathcal{A}\quad\text{if and only if}\quad 1\leqslant i\leqslant m'$$

where m' is less than or equal to m. We can easily check that $I\supset\{m'+1,\dots,m\}$ for any $I\in\mathcal{A}$ and $D_{m'+1},\dots,D_m$ are linearly independent over $\mathbb{R}$. The elements $D_1,\dots,D_m$ define the following exact sequence

$$0\longrightarrow\mathbb{L}\xrightarrow{(D_1,\dots,D_m)}\mathbb{Z}^m\xrightarrow{\beta}\boldsymbol{N}\longrightarrow 0,\tag{36}$$

where $\boldsymbol{N}$ is a finitely generated abelian group. By the long exact sequence associated with the functor $\mathrm{Tor}_\bullet(-,\mathbb{C}^*)$, we find that the torsion part $\boldsymbol{N}_{\mathrm{tor}}=\mathrm{Tor}_1(\boldsymbol{N},\mathbb{C}^*)$ of $\boldsymbol{N}$ is isomorphic to the generic stabilizer $\mathrm{Ker}(\mathbb{T}\to(\mathbb{C}^*)^m)$. The free part $\boldsymbol{N}_{\mathrm{free}}=\boldsymbol{N}/\boldsymbol{N}_{\mathrm{tor}}$ is of rank $n=\dim_{\mathbb{C}}\mathcal{X}$. Let $b_1,\dots,b_m$ be the images in $\boldsymbol{N}$ of the standard basis of $\mathbb{Z}^m$ under β. The *stacky fan* of $\mathcal{X}$, in the sense of Borisov, Chen, and Smith [12], is given by the following data:

- vectors $b_1,\dots,b_{m'}$ in $\boldsymbol{N}$;
- a complete simplicial fan Σ in $\boldsymbol{N}\otimes\mathbb{R}$ such that
  (i) the set of one-dimensional cones is $\{\mathbb{R}_{\geqslant 0}b_1,\dots,\mathbb{R}_{\geqslant 0}b_{m'}\}$;
  (ii) $\sigma_I=\sum_{i\notin I}\mathbb{R}_{\geqslant 0}b_i$ defines a cone of Σ if and only if $I\in\mathcal{A}$.

The toric variety defined by the fan Σ is the coarse moduli space of $\mathcal{X}$. The conditions (B) and (C) correspond to that Σ is simplicial and that Σ is complete, i.e. the union of all cones in Σ is $\boldsymbol{N}\otimes\mathbb{R}$. An element of $\mathcal{A}$ may be referred to as an "anticone."

Remark 3.1. Borisov, Chen, and Smith [12] defined a toric Deligne–Mumford stack starting from data of a stacky fan. Our construction can give every toric Deligne–Mumford stack in their sense which has a projective coarse moduli space. Note that the vectors $b_{m'+1},\dots,b_m$ do not appear as data of a stacky fan. The stacky fan together with these extra vectors gives an *extended stacky fan* in the sense of Jiang [52]. When we start from a stacky fan, our initial data can be given as the kernel of the map β by *choosing* extra vectors $b_{m'+1},\dots,b_m\in\boldsymbol{N}$ such that β is surjective. These redundant data allows us to define $\mathcal{X}$ as a quotient by a *connected* torus $\mathbb{T}$.

3.1.2. Kähler cone and a choice of a nef basis

Since every element of $\mathcal{A}$ contains $\{m'+1,\dots,m\}$, it is convenient to put

$$\mathcal{A}'=\bigl\{I'\subset\{1,\dots,m'\};\ I'\cup\{m'+1,\dots,m\}\in\mathcal{A}\bigr\}.$$

We can easily see that $\mathcal{U}_\eta$ factors as

$$\mathcal{U}_\eta=\mathcal{U}'_\eta\times(\mathbb{C}^*)^{m-m'},\quad \mathcal{U}'_\eta=\mathbb{C}^{m'}\Bigl\backslash\bigcup_{I'\notin\mathcal{A}'}\mathbb{C}^{I'}.$$

Thus we can write

$$\mathcal{X}=\bigl[\mathcal{U}'_\eta/\mathbb{G}\bigr],\quad \mathbb{G}:=\mathrm{Ker}\bigl(\mathbb{T}\to(\mathbb{C}^*)^m\to(\mathbb{C}^*)^{\{m'+1,\dots,m\}}\bigr).$$

Note that $\mathbb{G}$ is isomorphic to $(\mathbb{C}^*)^{r'}$ times a finite abelian group for $r' := r - (m - m')$. Every character $\xi : \mathbb{G} \to \mathbb{C}^*$ of $\mathbb{G}$ defines an orbifold line bundle $L_\xi := \mathcal{U}'_\eta \times_{\mathbb{G},\xi} \mathbb{C} \to \mathcal{X}$. Under this correspondence between ξ and L_ξ, the Picard group $\mathrm{Pic}(\mathcal{X})$ is identified with the character group $\mathrm{Hom}(\mathbb{G}, \mathbb{C}^*)$ and also with $H^2(\mathcal{X}, \mathbb{Z})$ (via c_1):

$$\mathrm{Pic}(\mathcal{X}) \cong \mathrm{Hom}(\mathbb{G}, \mathbb{C}^*) \cong \mathbb{L}^\vee \Big/ \sum_{i=m'+1}^{m} \mathbb{Z} D_i \cong H^2(\mathcal{X}, \mathbb{Z}).$$

The image $\overline{D}_i$ of D_i in $H^2(\mathcal{X}, \mathbb{R})$ is the Poincaré dual of the toric divisor $\{z_i = 0\} \subset \mathcal{X}$ for $1 \leqslant i \leqslant m'$. Over rational numbers, we have

$$H^2(\mathcal{X}, \mathbb{Q}) \cong \mathbb{L}^\vee \otimes \mathbb{Q} \Big/ \sum_{i=m'+1}^{m} \mathbb{Q} D_i,$$

$$H_2(\mathcal{X}, \mathbb{Q}) \cong \mathrm{Ker}\big((D_{m'+1}, \ldots, D_m) : \mathbb{L} \otimes \mathbb{Q} \to \mathbb{Q}^{m-m'}\big) \subset \mathbb{L} \otimes \mathbb{Q}.$$

Now we introduce a canonical splitting (over $\mathbb{Q}$) of the surjection $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X}, \mathbb{Q})$. For $m' < j \leqslant m$, b_j is contained in some cone in Σ since Σ is complete. Namely,

$$b_j = \sum_{i \notin I_j} c_{ji} b_i, \quad \text{in } \boldsymbol{N} \otimes \mathbb{Q},\ c_{ji} \geqslant 0,\ \exists I_j \in \mathcal{A}, \tag{37}$$

where I_j is the "anticone" of the cone containing b_j. By the exact sequence (36) tensored with $\mathbb{Q}$, we can find $D_j^\vee \in \mathbb{L} \otimes \mathbb{Q}$ such that

$$\langle D_i, D_j^\vee \rangle = \begin{cases} 1 & i = j, \\ -c_{ji} & i \notin I_j, \\ 0 & i \in I_j \setminus \{j\}. \end{cases}$$

Note that $D_j^\vee$ is uniquely determined by these conditions. These vectors $D_j^\vee$ define a decomposition

$$\mathbb{L}^\vee \otimes \mathbb{Q} = \mathrm{Ker}\big(\big(D_{m'+1}^\vee, \ldots, D_m^\vee\big) : \mathbb{L}^\vee \otimes \mathbb{Q} \to \mathbb{Q}^{m-m'}\big) \oplus \bigoplus_{j=m'+1}^{m} \mathbb{Q} D_j. \tag{38}$$

The first factor $\mathrm{Ker}(D_{m'+1}^\vee, \ldots, D_m^\vee)$ is identified with $H^2(\mathcal{X}, \mathbb{Q})$ under the surjection $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X}, \mathbb{Q})$. Via this decomposition, we henceforth regard $H^2(\mathcal{X}, \mathbb{Q})$ as a subspace of $\mathbb{L}^\vee \otimes \mathbb{Q}$. We define an *extended Kähler cone* $\widetilde{C}_{\mathcal{X}}$ as

$$\widetilde{C}_{\mathcal{X}} = \bigcap_{I \in \mathcal{A}} \Big(\sum_{i \in I} \mathbb{R}_{>0} D_i \Big) \subset \mathbb{L}^\vee \otimes \mathbb{R}.$$

Then $\eta \in \widetilde{C}_{\mathcal{X}}$ and the image of η in $H^2(\mathcal{X}, \mathbb{R})$ is the class of the reduced symplectic form. The set $\widetilde{C}_{\mathcal{X}}$ is the connected component of the set of regular values of the moment map $\mathfrak{h} : \mathbb{C}^m \to \mathbb{L}^\vee \otimes \mathbb{R}$, which contains η. The extended Kähler cone depends not only on $\mathcal{X}$ but also on the choice of our initial data. The genuine *Kähler cone* $C_{\mathcal{X}}$ of $\mathcal{X}$ is the image of $\widetilde{C}_{\mathcal{X}}$ under $\mathbb{L}^\vee \otimes \mathbb{R} \to H^2(\mathcal{X}, \mathbb{R})$:

$$C_{\mathcal{X}} = \bigcap_{I' \in \mathcal{A}'} \left(\sum_{i \in I'} \mathbb{R}_{>0} \overline{D}_i \right) \subset H^2(\mathcal{X}, \mathbb{R}) = H^{1,1}(\mathcal{X}, \mathbb{R})$$

where $\overline{D}_i$ is the image of D_i in $H^2(\mathcal{X}, \mathbb{R})$. The next lemma means that the extended Kähler cone also "splits."

Lemma 3.2. $\widetilde{C}_{\mathcal{X}} = C_{\mathcal{X}} + \sum_{j=m'+1}^{m} \mathbb{R}_{>0} D_j$ *in* $\mathbb{L}^\vee \otimes \mathbb{R} \cong H^2(\mathcal{X}, \mathbb{R}) \oplus \bigoplus_{j=m'+1}^{m} \mathbb{R} D_j$.

Proof. First note that for $1 \leqslant i \leqslant m'$, $\overline{D}_i = D_i + \sum_{j>m'} c_{ji} D_j$, where $c_{ji} = -\langle D_i, D_j^\vee \rangle \geqslant 0$. Take $I' \in \mathcal{A}'$ and put $I = I' \cup \{m'+1, \ldots, m\}$. It is easy to check that

$$\sum_{i \in I'} \mathbb{R}_{>0} \overline{D}_i + \sum_{j=m'+1}^{m} \mathbb{R}_{>0} D_j = \sum_{k \in I} \mathbb{R}_{>0} D_k \cap \bigcap_{j=m'+1}^{m} \{ D_j^\vee > 0 \},$$

where we regard $D_j^\vee$ as a linear function on $\mathbb{L}^\vee \otimes \mathbb{R}$. Thus $C_{\mathcal{X}} + \sum_{j>m'} \mathbb{R}_{>0} D_j = \widetilde{C}_{\mathcal{X}} \cap \bigcap_{j=m'+1}^{m} \{D_j^\vee > 0\}$. For $j > m'$, take $I_j \in \mathcal{A}$ appearing in (37). Then $\widetilde{C}_{\mathcal{X}} \subset \sum_{k \in I_j} \mathbb{R}_{>0} D_k \subset \{D_j^\vee > 0\}$. The conclusion follows. $\square$

We choose an integral basis $\{p_1, \ldots, p_r\}$ of $\mathbb{L}^\vee$ such that p_a is in the closure $\mathrm{cl}(\widetilde{C}_{\mathcal{X}})$ of $\widetilde{C}_{\mathcal{X}}$ for all a and $p_{r'+1}, \ldots, p_r$ are in $\sum_{i=m'+1}^{m} \mathbb{R}_{\geqslant 0} D_i$. Since the decomposition (38) is defined over $\mathbb{Q}$, it is not always possible to choose $p_1, \ldots, p_{r'}$ from $\mathrm{cl}(C_{\mathcal{X}})$. The images $\overline{p}_1, \ldots, \overline{p}_{r'}$ of $p_1, \ldots, p_{r'}$ in $H^2(\mathcal{X}, \mathbb{R})$ are nef and those of $p_{r'+1}, \ldots, p_r$ are zero. Define a matrix (m_{ia}) by

$$D_i = \sum_{a=1}^{r} \mathsf{m}_{ia} p_a, \quad \mathsf{m}_{ia} \in \mathbb{Z}. \tag{39}$$

Then the class $\overline{D}_i$ of the toric divisor $\{z_i = 0\}$ is given by

$$\overline{D}_i = \sum_{a=1}^{r'} \mathsf{m}_{ia} \overline{p}_a. \tag{40}$$

Then $\overline{D}_j = 0$ for $m' < j \leqslant m$.

3.1.3. Inertia components and orbifold cohomology

We introduce subsets $\mathbb{K}$, $\mathbb{K}_{\mathrm{eff}}$ of $\mathbb{L} \otimes \mathbb{Q}$ by

$$\mathbb{K} = \big\{ d \in \mathbb{L} \otimes \mathbb{Q};\ \big\{ i \in \{1, \ldots, m\};\ \langle D_i, d \rangle \in \mathbb{Z} \big\} \in \mathcal{A} \big\},$$
$$\mathbb{K}_{\mathrm{eff}} = \big\{ d \in \mathbb{L} \otimes \mathbb{Q};\ \big\{ i \in \{1, \ldots, m\};\ \langle D_i, d \rangle \in \mathbb{Z}_{\geqslant 0} \big\} \in \mathcal{A} \big\}.$$

The sets $\mathbb{K}$ and $\mathbb{K}_{\mathrm{eff}}$ are not closed under addition, but $\mathbb{L}$ acts on $\mathbb{K}$. The set $\mathbb{K}_{\mathrm{eff}} \cap H_2(\mathcal{X}, \mathbb{R})$ consists of classes of stable maps from $\mathbb{P}(1, a)$ to $\mathcal{X}$ for some $a \in \mathbb{N}$. It follows from the definition that $\mathbb{K}_{\mathrm{eff}}$ pairs with $\widetilde{C}_{\mathcal{X}}$ positively. The index set T of components of the inertia stack $I\mathcal{X}$ is given by Box [12]:

$$\mathrm{Box} := \left\{ v \in \boldsymbol{N};\ v = \sum_{k \notin I} c_k b_k \text{ in } \boldsymbol{N} \otimes \mathbb{Q},\ c_k \in [0,1),\ I \in \mathcal{A} \right\}.$$

For a real number r, let $\lceil r \rceil$, $\lfloor r \rfloor$ and $\{r\}$ denote the ceiling, floor and fractional part of r respectively. For $d \in \mathbb{K}$, we define $v(d) \in \mathrm{Box}$ by

$$v(d) := \sum_{i=1}^{m} \lceil \langle D_i, d \rangle \rceil b_i \in \boldsymbol{N}.$$

Note that $v(d)$ belongs to Box because

$$v(d) = \sum_{i=1}^{m} \big(\{ -\langle D_i, d \rangle \} + \langle D_i, d \rangle \big) b_i = \sum_{i=1}^{m} \{ -\langle D_i, d \rangle \} b_i \quad \text{in } \boldsymbol{N} \otimes \mathbb{Q}$$

by the exact sequence (36). This map $d \mapsto v(d)$ factors through $\mathbb{K} \to \mathbb{K}/\mathbb{L}$ and identifies $\mathbb{K}/\mathbb{L}$ with Box. The corresponding inertia component[3] $\mathcal{X}_{v(d)}$ is defined by

$$\mathcal{X}_{v(d)} := \big\{ [z_1, \ldots, z_m] \in \mathcal{X};\ z_i = 0 \text{ if } \langle D_i, d \rangle \notin \mathbb{Z} \big\}.$$

The stabilizer along $\mathcal{X}_{v(d)}$ is defined to be $\exp(-2\pi\sqrt{-1}d) \in \mathbb{L} \otimes \mathbb{C}^* \cong \mathbb{T}$, which acts on $\mathbb{C}^m$ by

$$\big(e^{-2\pi \mathbf{i} \langle D_1, d \rangle}, \ldots, e^{-2\pi \mathbf{i} \langle D_m, d \rangle} \big).$$

It is easy to check that $\mathcal{X}_{v(d)}$ depends only on the element $v(d) \in \mathrm{Box}$. The age of $\mathcal{X}_{v(d)}$ is given by

$$\iota_{v(d)} = \sum_{i=1}^{m} \{ -\langle D_i, d \rangle \} = \sum_{i=1}^{m'} \{ -\langle D_i, d \rangle \}. \tag{41}$$

The inertia stack and orbifold cohomology are given by

$$I\mathcal{X} = \bigsqcup_{v \in \mathrm{Box}} \mathcal{X}_v, \qquad H^i_{\mathrm{orb}}(\mathcal{X}) = \bigoplus_{v \in \mathrm{Box}} H^{i - 2\iota_v}(\mathcal{X}_v). \tag{42}$$

Denote by $\mathbf{1}_v$ the unit class of $H^*(\mathcal{X}_v)$. Each inertia component $\mathcal{X}_v$ is again a toric orbifold and its cohomology ring is generated by the degree two classes $\overline{p}_1, \ldots, \overline{p}_{r'}$:

$$H^*(\mathcal{X}_{v(d)}) = \mathbb{C}[\overline{p}_1, \ldots, \overline{p}_{r'}] \mathbf{1}_v \cong \mathbb{C}[\overline{p}_1, \ldots, \overline{p}_{r'}] / \mathfrak{J}_{v(d)},$$
$$\text{where } \mathfrak{J}_{v(d)} := \left\langle \prod_{i \in I} \overline{D}_i;\ \big\{ 1 \leqslant i \leqslant m;\ \langle D_i, d \rangle \in \mathbb{Z} \big\} \setminus I \notin \mathcal{A} \right\rangle. \tag{43}$$

[3] When $d \in \mathbb{K}_{\mathrm{eff}} \cap H_2(\mathcal{X}, \mathbb{Q})$, the evaluation image of a stable map $\mathbb{P}(1,a) \to \mathcal{X}$ of degree d at the stacky marked point $\mathbb{P}(a) \in \mathbb{P}(1,a)$ lies in $\mathcal{X}_{\mathrm{inv}(v(d))}$.

Here we regard $\overline{D}_i$ as a linear form (40) in $\overline{p}_a$. For $\xi \in \mathbb{L}^\vee$, let $[\xi]$ be the image of ξ in $\mathbb{L}^\vee / \sum_{j=m'+1}^m \mathbb{Z}D_j \cong H^2(\mathcal{X}, \mathbb{Z})$. The age $f_v(\xi) = f_v([\xi]) \in [0,1)$ of the line bundle L_ξ (see Section 2.2) is given by

$$f_{v(d)}(\xi) = \{-\langle \xi, d\rangle\}, \quad d \in \mathbb{K}. \tag{44}$$

3.1.4. Weak Fano condition

The first Chern class $\rho = c_1(T\mathcal{X}) \in H^2(\mathcal{X}, \mathbb{Q})$ of $\mathcal{X}$ is the image of the vector $\hat{\rho} \in \mathbb{L}^\vee$:

$$\hat{\rho} := D_1 + \cdots + D_m = \sum_{a=1}^r \rho_a p_a, \quad \rho_a := \sum_{i=1}^m \mathsf{m}_{ia}.$$

We call $\mathcal{X}$ *weak Fano* if ρ is in the closure $\mathrm{cl}(C_\mathcal{X})$ of the Kähler cone $C_\mathcal{X}$. We shall need a little stronger condition $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_\mathcal{X})$. This condition $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_\mathcal{X})$ depends not only on $\mathcal{X}$ but also on our initial data in Section 3.1.1, i.e. the choice of the vectors $b_{m'+1}, \ldots, b_m \in \boldsymbol{N}$.

Lemma 3.3. *We have $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_\mathcal{X})$ if and only if $\rho \in \mathrm{cl}(C_\mathcal{X})$ (i.e. $\mathcal{X}$ is weak Fano) and* $\mathrm{age}(b_j) := \sum_{i \notin I_j} c_{ji} \leqslant 1$ *for all $j > m'$. If $b_j \in$* Box, $\mathrm{age}(b_j)$ *coincides with ι_{b_j} in* (41)*; see* (37) *for the definition of I_j and c_{ji}.*

Proof. From $\overline{D}_i = D_i + \sum_{j>m'} c_{ji} D_j$, we have

$$\hat{\rho} = \rho + \sum_{j>m'} \big(1 - \mathrm{age}(b_j)\big) D_j.$$

The conclusion follows from Lemma 3.2. □

When $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_\mathcal{X})$, we can choose a basis $p_1, \ldots, p_r \in \mathrm{cl}(\widetilde{C}_\mathcal{X})$ so that $\hat{\rho}$ is in the cone generated by p_a's. Thus *in this case, we can assume $\rho_a \geqslant 0$ without loss of generality.*

Remark 3.4. The condition $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_\mathcal{X})$ depends on the choice of our initial data. This can be achieved if $\mathcal{X}$ is weak Fano and if in addition its stacky fan satisfies

$$\big\{v \in \mathrm{Box};\ \mathrm{age}(v) \leqslant 1\big\} \cup \{b_1, \ldots, b_{m'}\} \quad \text{generates } \boldsymbol{N} \text{ over } \mathbb{Z}.$$

If this holds, we can choose $b_{m'+1}, \ldots, b_m \in \mathrm{Box}$ so that $\{b_1, \ldots, b_m\}$ generates $\boldsymbol{N}$ and $\mathrm{age}(b_j) \leqslant 1$ for $m' < j \leqslant m$. Then the exact sequence (36) determines $D_1, \ldots, D_m$ and $\hat{\rho} = D_1 + \cdots + D_m \in \mathrm{cl}(\widetilde{C}_\mathcal{X})$ holds. If $\mathcal{X}$ is simply-connected in the sense of orbifold ($\pi_1^{\mathrm{orb}}(\mathcal{X}) = \mathbf{1}$), $\boldsymbol{N}$ is generated by $b_1, \ldots, b_{m'}$.

Remark 3.5. The vectors D_j, $m' < j \leqslant m$ in $\mathbb{L}^\vee$ correspond to the following elements in the twisted sector:

$$\mathfrak{D}_j = \prod_{i \notin I_j} \overline{D}_i^{\lfloor c_{ji} \rfloor} \mathbf{1}_{v(D_j^\vee)} \in H^*_{\mathrm{orb}}(\mathcal{X}), \quad \text{where } v\big(D_j^\vee\big) = b_j + \sum_{i \notin I_j} \lceil -c_{ji} \rceil b_i. \tag{45}$$

This correspondence can be seen from the expansion (60) of the mirror map $\tau(q)$ below. We have $\mathfrak{D}_j = \mathbf{1}_{b_j}$ when $b_j \in \mathrm{Box}$. Therefore, if $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$ and $b_{m'+1}, \ldots, b_m$ are mutually different elements in Box, we can identify $\mathbb{L}^\vee \otimes \mathbb{C}$ with the subspace $H^2(\mathcal{X}) \oplus \bigoplus_{j>m'} H^0(\mathcal{X}_{b_j})$ of $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$.

3.2. *Landau–Ginzburg model*

We introduce the Landau–Ginzburg (LG) model mirror to compact toric orbifolds. We use the notation from Section 3.1.

3.2.1. *Definition*

By applying the exact functor $\mathrm{Hom}(-, \mathbb{C}^*)$ to the short exact sequence (36), we have

$$\mathbf{1} \longrightarrow \mathrm{Hom}(\boldsymbol{N}, \mathbb{C}^*) \longrightarrow Y := (\mathbb{C}^*)^m \xrightarrow{\mathrm{pr}} \mathcal{M} := \mathrm{Hom}(\mathbb{L}, \mathbb{C}^*) \longrightarrow \mathbf{1}. \tag{46}$$

The *Landau–Ginzburg model* (LG model for short) associated to a toric orbifold is the family $\mathrm{pr}: Y \to \mathcal{M}$ of affine varieties given by the third arrow and a fiberwise Laurent polynomial $W: Y \to \mathbb{C}$, called potential, given by

$$W = w_1 + \cdots + w_m$$

where $w_1, \ldots, w_m$ are the standard $\mathbb{C}^*$-valued co-ordinates on $Y = (\mathbb{C}^*)^m$. Roughly speaking, the base space $\mathcal{M} = \mathbb{L}^\vee \otimes \mathbb{C}^*$ corresponds to the extended (and complexified) Kähler moduli space $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$ of $\mathcal{X}$ under mirror symmetry (see Remark 3.5). The basis of $\mathbb{L}$ dual to $p_1, \ldots, p_r \in \mathbb{L}^\vee$ in Section 3.1.2 defines $\mathbb{C}^*$-valued co-ordinates $q_1, \ldots, q_r$ on $\mathcal{M} = \mathrm{Hom}(\mathbb{L}, \mathbb{C}^*)$. Then the projection is given by (see (39))

$$\mathrm{pr}(w_1, \ldots, w_m) = (q_1, \ldots, q_r), \quad q_a = \prod_{i=1}^m w_i^{\mathsf{m}_{ia}}. \tag{47}$$

Let $Y_q := \mathrm{pr}^{-1}(q)$ be the fiber at $q \in \mathcal{M}$ and set $W_q := W|_{Y_q}$. Note that Y_q has $|\boldsymbol{N}_{\mathrm{tor}}|$ connected components and each connected component is isomorphic to $\mathrm{Hom}(\boldsymbol{N}_{\mathrm{free}}, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. Let $e_1, \ldots, e_n$ be an arbitrary basis of $\boldsymbol{N}_{\mathrm{free}}$ and $y_1, \ldots, y_n$ be the corresponding $\mathbb{C}^*$-valued co-ordinate on $\mathrm{Hom}(\boldsymbol{N}_{\mathrm{free}}, \mathbb{C}^*)$. We choose a splitting of the exact sequence dual to (36) over rational numbers. Namely, we take a matrix $(\ell_{ia})_{1\leqslant i\leqslant m, 1\leqslant a\leqslant r}$ with $\ell_{ia} \in \mathbb{Q}$ such that $p_a = \sum_{i=1}^m D_i \ell_{ia}$. This splitting defines a multi-valued section of $\mathrm{pr}: Y \to \mathcal{M}$ and identifies Y_q with $\mathrm{Hom}(\boldsymbol{N}, \mathbb{C}^*)$. Under this identification, $y_1, \ldots, y_n$ give co-ordinates on each connected component of Y_q and we have

$$W|_{Y_q} = W_q = q^{\ell_1} y^{b_1} + \cdots + q^{\ell_m} y^{b_m}, \quad q^{\ell_i} = \prod_{a=1}^r q_a^{\ell_{ia}}, \ y^{b_i} = \prod_{j=1}^n y_j^{b_{ij}}, \tag{48}$$

where $b_i = \sum_{j=1}^n b_{ij} e_j$ in $\boldsymbol{N}_{\mathrm{free}}$. Here, the choice of the branches of fractional powers of q_a appearing in q^{ℓ_i} depends on a connected component of Y_q.

3.2.2. Kouchnirenko's condition

When constructing the B-model D-module, we shall need to restrict the parameter $q \in \mathcal{M}$ to some Zariski open subset $\mathcal{M}^{\mathrm{o}} \subset \mathcal{M}$ so that W_q satisfies the "non-degeneracy condition at infinity" due to Kouchnirenko [56, 1.19].

Definition 3.6. Let $\widehat{S}$ denote the convex hull of $b_1, \dots, b_m \in \boldsymbol{N} \otimes \mathbb{R}$. We call the Laurent polynomial $W_q(y)$ of the form (48) *non-degenerate at infinity* if for every face Δ of $\widehat{S}$ (where $0 \leqslant \dim \Delta \leqslant n-1$), $W_{q,\Delta}(y) := \sum_{b_i \in \Delta} q^{\ell_i} y^{b_i}$ does not have critical points on $y \in (\mathbb{C}^*)^n$. Let $\mathcal{M}^{\mathrm{o}}$ be the subset of $\mathcal{M}$ consisting of q for which W_q is non-degenerate at infinity.

Proposition 3.7.

- (i) *Under the condition* (C) *in Section* 3.1.1, $0 \in \boldsymbol{N} \otimes \mathbb{R}$ *is in the interior of* $\widehat{S}$. *Therefore, the Laurent polynomial* W_q *is* convenient *in the sense of Kouchnirenko* [56, 1.5].
- (ii) $\mathcal{M}^{\mathrm{o}}$ *is an open and dense subset of* $\mathcal{M}$ *in Zariski topology.*
- (iii) *For* $q \in \mathcal{M}^{\mathrm{o}}$, $W_q(y)$ *has* $|\boldsymbol{N}_{\mathrm{tor}}| \times n! \operatorname{Vol}(\widehat{S})$ *critical points on* Y_q *(counted with multiplicities).*

Proof. The condition (C) implies that there exists $d \in \mathbb{L}$ such that $c_i := \langle D_i, d\rangle > 0$. Then by the exact sequence (36), we have $\sum_{i=1}^m c_i b_i = 0$. This proves (i). The statements (ii) and (iii) are due to Kouchnirenko. (ii) follows from (i) and the same argument as in [56, 6.3]. One of main theorems in [56, 1.16] states that $W_q(y)$ has $n! \operatorname{Vol}(\widehat{S})$ number of critical points on each connected component of Y_q. (iii) follows from this and $|\pi_0(Y_q)| = |\boldsymbol{N}_{\mathrm{tor}}|$. $\square$

The following lemma shows that the Kouchnirenko's condition holds on a certain "cylindrical end" of $\mathcal{M}$. A proof is given in Appendix A.1.

Lemma 3.8. *Let* $q_1, \dots, q_r$ *be the co-ordinates on* $\mathcal{M}$ *dual to the basis* $p_1, \dots, p_r \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$ *chosen in Section* 3.1.2. *There exists* $\epsilon > 0$ *such that* $q \in \mathcal{M}^{\mathrm{o}}$ *if* $0 < |q_a| < \epsilon$ *for all* a.

Lemma 3.9. *Assume that* $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$. *Then* $\widehat{S}$ *is the union of simplices* $\widehat{S}(\sigma) := \{\sum_{b_i \in \sigma} c_i b_i;\ c_i \in [0,1], \sum_{b_i \in \sigma} c_i \leqslant 1\}$ *over maximal (n-dimensional) cones* σ *of the fan* Σ *of* $\mathcal{X}$. *Moreover, we have* $|\boldsymbol{N}_{\mathrm{tor}}| \times n! \operatorname{Vol}(\widehat{S}) = \dim H^*_{\mathrm{orb}}(\mathcal{X})$.

Proof. By Lemma 3.3, $\rho = c_1(\mathcal{X})$ is nef. This implies that the piecewise linear function $h : \boldsymbol{N} \otimes \mathbb{R} \to \mathbb{R}$ on the fan Σ (linear on each maximal cone in Σ) defined by $h(b_i) = 1$ for $1 \leqslant i \leqslant m'$ is convex (see [61]). Therefore, $\bigcup_{\sigma:\ \dim\sigma = n} \widehat{S}(\sigma) = h^{-1}((-\infty, 1])$ is convex. Because b_j, $j > m'$ is contained in this by Lemma 3.3, we have $\widehat{S} = \bigcup_{\sigma:\ \dim\sigma = n} \widehat{S}(\sigma)$.

Because odd cohomology groups of $\mathcal{X}_v$ vanish, $\dim H^*(\mathcal{X}_v)$ is equal to the Euler number of $\mathcal{X}_v$, so is equal to the number of torus fixed points on $\mathcal{X}_v$ (for the natural torus action) by Poincaré–Hopf. Torus fixed points on $\mathcal{X}_v$ are parametrized by maximal cones σ in the fan Σ such that σ contains the image of $v \in \mathrm{Box}$ in $\boldsymbol{N} \otimes \mathbb{R}$. Hence,

$$\sum_{v \in \mathrm{Box}} \dim H^*(\mathcal{X}_v) = \sum_{\sigma:\ \dim\sigma = n} \sharp\{v \in \mathrm{Box};\ v \in \sigma\} = \sum_{\sigma:\ \dim\sigma = n} |\boldsymbol{N}_{\mathrm{tor}}| \times n! \operatorname{Vol}\bigl(\widehat{S}(\sigma)\bigr).$$

The conclusion follows. $\square$

3.2.3. Jacobi ring and Batyrev ring

The *Jacobi ring* $J(W)$ is the ring of functions on the (fiberwise) critical set of W:

$$J(W) := \mathbb{C}\big[w_1^{\pm}, \dots, w_m^{\pm}\big] \Big/ \Big\langle y_1 \frac{\partial W}{\partial y_1}, \dots, y_n \frac{\partial W}{\partial y_n} \Big\rangle.$$

Note that $J(W)$ is a $\mathbb{C}[q^{\pm}] := \mathbb{C}[q_1^{\pm}, \dots, q_r^{\pm}]$-algebra. Denote by $J(W_q) = J(W) \otimes_{\mathbb{C}[q^{\pm}]} \mathbb{C}_q$ the fiber of $J(W)$ at $q \in \mathcal{M} = \operatorname{Spec} \mathbb{C}[q^{\pm}]$. By Proposition 3.7, $J(W_q)$ is of dimension $|\boldsymbol{N}_{\mathrm{tor}}| \times n! \operatorname{Vol}(\widehat{S})$ when $q \in \mathcal{M}^{\mathrm{o}}$. The *Batyrev ring* is defined by

$$B(\mathcal{X}) := \mathbb{C}\big[q^{\pm}\big][\mathsf{p}_1, \dots, \mathsf{p}_r] \Big/ \Big\langle q^d \prod_{i:\ \langle D_i, d\rangle < 0} \mathsf{w}_i^{-\langle D_i, d\rangle} - \prod_{i:\ \langle D_i, d\rangle > 0} \mathsf{w}_i^{\langle D_i, d\rangle};\ d \in \mathbb{L} \Big\rangle$$

where $q^d := \prod_{a=1}^r q_a^{\langle p_a, d\rangle}$ and $\mathsf{w}_i := \sum_{a=1}^r \mathsf{m}_{ia} \mathsf{p}_a$. By the condition (C) in Section 3.1.1, there exists $d \in \mathbb{L}$ such that $c_i := \langle D_i, d\rangle > 0$ for all i. Hence $\prod_{i=1}^m \mathsf{w}_i^{c_i} = q^d$ holds in $B(\mathcal{X})$ and therefore w_i is invertible in $B(\mathcal{X})$. With this fact in mind, Batyrev ring is given by the simple relations (note that m_{ia} can be negative)

$$q_a = \prod_{i=1}^m \mathsf{w}_i^{\mathsf{m}_{ia}} = \prod_{i=1}^m \Bigg(\sum_{b=1}^r \mathsf{m}_{ib} \mathsf{p}_b \Bigg)^{\mathsf{m}_{ia}}, \quad 1 \leqslant a \leqslant r. \tag{49}$$

The following was shown in [47, Lemma 5.10, Proposition 5.11] for toric manifolds.

Proposition 3.10.

(i) *The map $B(\mathcal{X}) \to J(W)$, $\mathsf{p}_a \mapsto [q_a(\partial W_q/\partial q_a)]$ defines an isomorphism of $\mathbb{C}[q^{\pm}]$-algebras.*
(ii) *Let $\mathcal{M}^{\mathrm{oo}}$ be the subset of $\mathcal{M}^{\mathrm{o}}$ consisting of $q \in \mathcal{M}^{\mathrm{o}}$ such that all the critical points of W_q are non-degenerate. Then $\mathcal{M}^{\mathrm{oo}}$ is open and dense in $\mathcal{M}^{\mathrm{o}}$.*

Proof. (i) Since $\mathrm{pr}_*(w_i(\partial/\partial w_i)) = \sum_{a=1}^r \mathsf{m}_{ia} q_a(\partial/\partial q_a)$, we have

$$\Bigg[\sum_{a=1}^r \mathsf{m}_{ia} q_a \frac{\partial W_q}{\partial q_a} \Bigg] = \Bigg[w_i \frac{\partial W}{\partial w_i} \Bigg] = [w_i] \quad \text{in } J(W).$$

This shows that w_i maps to an invertible element $[w_i] \in J(W)$ satisfying $\prod_{i=1}^m [w_i]^{\mathsf{m}_{ia}} = q_a$. Thus the map $B(\mathcal{X}) \to J(W)$ is well-defined. The inverse map, sending $[w_i]$ to w_i, is also well-defined. The details are left to the reader.

(ii) The isomorphism in (i) induces an isomorphism $\operatorname{Spec} B(\mathcal{X}) \cong \operatorname{Spec} J(W)$ over $\mathcal{M}$. Since $\operatorname{Spec} B(\mathcal{X})$ can be written as the graph of the map $\mathsf{p} \mapsto q$ (49), it suffices to show that this map is a local isomorphism at generic p. This follows from the fact that the Jacobian $\partial \log q_a / \partial \mathsf{p}_b = \sum_{i=1}^m \mathsf{m}_{ia} \mathsf{w}_i^{-1} \mathsf{m}_{ib}$ of the map (49) is positive definite when $\mathsf{w}_i > 0$. (Note that we can choose $\mathsf{p}_b \in \mathbb{R}$ so that $\mathsf{w}_i = \sum_{b=1}^r \mathsf{m}_{ib} \mathsf{p}_b > 0$ for all i again by the condition (C).) $\square$

3.3. B-model D-module

Here we describe the B-model D-module in two steps. First we construct a local system over $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ using the Morse theory for $\Re(W_q/z)$. Then we extend the local system to a meromorphic flat connection over $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}$ using de Rham forms and oscillatory integrals.

3.3.1. Local system of Lefschetz thimbles

Let $f_{q,z}: Y_q \to \mathbb{R}$ be the real part of the function $y \mapsto W_q(y)/z$. The following lemma allows us to use Morse theory for the improper function $f_{q,z}(y)$.

Lemma 3.11. *For each $\epsilon > 0$, the family of topological spaces*

$$\bigcup_{(q,z)\in\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*} \left\{y \in Y_q;\ \left\|df_{q,z}(y)\right\| \leqslant \epsilon\right\} \to \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$$

is proper, i.e. pull-back of a compact set is compact. Here the norm $\|df_{q,z}(y)\|$ is taken with respect to the complete Kähler metric $\frac{1}{\mathbf{i}}\sum_{i=1}^n d\log y_i \wedge d\,\overline{\log y_i}$ on Y_q.

A similar result for polynomial functions can be found in [64, Proposition 2.2 and Remarque] and this lemma may be well-known to specialists. A proof is given in Appendix A.2 since the author was not able to find a suitable reference. Lemma 3.11 implies that $f_{q,z}$ satisfies the Palais–Smale condition, so that usual Morse theory applies to $f_{q,z}$ (see e.g. [62]). Take $(q,z) \in \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$. Since the set $\{y \in Y_q;\ \|df_{q,z}(y)\| < \epsilon\}$ is compact, we can choose $M \ll 0$ so that this set is contained in $\{y \in Y_q;\ f_{q,z}(y) > M\}$. Then the relative homology group $H_n(Y_q, \{y \in Y_q;\ f_{q,z}(y) \leqslant M\}; \mathbb{Z})$ is independent of the choice of such M and we denote this by

$$R^\vee_{\mathbb{Z},(q,z)} = H_n\big(Y_q, \left\{y \in Y_q;\ f_{q,z}(y) \ll 0\right\}; \mathbb{Z}\big), \quad (q,z) \in \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*. \tag{50}$$

The number of critical points of $f_{q,z}(y)$ is $N := |\boldsymbol{N}_{\mathrm{tor}}| \times n!\,\mathrm{Vol}(\widehat{S})$ by Proposition 3.7. If all the critical points of $W_q(y)$ are non-degenerate, by the standard argument in Morse theory, we know that Y_q is obtained from $\{f_{q,z}(y) \leqslant M\}$ by attaching N n-handles and so $R^\vee_{\mathbb{Z},(q,z)}$ is a free abelian group of rank N. If $W_q(y)$ has a critical point y_0 of multiplicity $\mu_0 > 1$, one can find[4] a small C^∞-perturbation $\tilde{f}_{q,z}$ of $f_{q,z}$ on a small neighborhood U_0 of y_0 such that $\tilde{f}_{q,z}$ has just μ_0 non-degenerate critical points in U_0 with Morse index n. By considering such a perturbation and Morse theory for $f_{q,z}$ in families (parametrized by q and z), we obtain the following.

Proposition 3.12. *The relative homology groups $R^\vee_{\mathbb{Z},(q,z)}$ in* (50) *form a local system of rank $|\boldsymbol{N}_{\mathrm{tor}}| \times n!\,\mathrm{Vol}(\widehat{S})$ over $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$.*

When all the critical points $\mathrm{cr}_1, \ldots, \mathrm{cr}_N$ of $W_q: Y_q \to \mathbb{C}$ are non-degenerate, a basis of the local system $R^\vee_{\mathbb{Z}}$ is given by a set of *Lefschetz thimbles* $\Gamma_1, \ldots, \Gamma_N$: the image of Γ_i under W_q/z

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[4] We can find $\tilde{f}_{q,z}$ in the following way: Let $\rho: \mathbb{R}_{\geqslant 0} \to [0,1]$ be a C^∞-function such that $\rho(r) = 1$ for $0 \leqslant r \leqslant 1/2$ and $\rho(r) = 0$ for $r \geqslant 1$. Let U_0 be an ϵ-neighborhood of y_0 (in the above Kähler metric) which does not contain other critical points. Let $t = (t_1, \ldots, t_n)$ be co-ordinates given by $y_i = y_{0,i}e^{t_i}$. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, put $f^a_{q,z}(y) = f_{q,z}(y) + \rho(|t|/\epsilon)\Re(at)$. Then for a generic, sufficiently small a, $\tilde{f}_{q,z} = f^a_{q,z}$ satisfies the conditions above (here, new critical points are all in $|t| < \epsilon/2$).

is given by a curve $\gamma_i : [0, \infty) \to \mathbb{C}$ such that $\gamma(0) = W_q(\mathrm{cr}_i)/z$, that $\Re\gamma_i(t)$ decreases monotonically to $-\infty$ as $t \to \infty$ and that γ_i does not pass through critical values other than $W_q(\mathrm{cr}_i)/z$; Γ_i is the union of cycles in $W_q^{-1}(z\gamma_i(t))$ collapsing to cr_i along the path $\gamma_i(t)$ as $t \to 0$. When the imaginary parts $\Im(W_q(\mathrm{cr}_1)/z), \dots, \Im(W_q(\mathrm{cr}_N)/z)$ are mutually different, Γ_i can be taken to be the union of downward gradient flowlines of $f_{q,z}(y)$ emanating from cr_i. (Note that the gradient flow of $f_{q,z} = \Re(W_q/z)$ with respect to a Kähler metric coincides with the Hamiltonian flow generated by $\Im(W_q/z)$.) Then γ_i becomes a half-line parallel to the real axis. The intersection pairing defines a unimodular pairing:

$$R^{\vee}_{\mathbb{Z},(q,-z)} \times R^{\vee}_{\mathbb{Z},(q,z)} \to \mathbb{Z}. \tag{51}$$

Let $R_{\mathbb{Z}} \to \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ be the local system dual to $R^{\vee}_{\mathbb{Z}}$ and $\mathcal{R} := R_{\mathbb{Z}} \otimes \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$ be the associated locally free sheaf on $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$. The sheaf $\mathcal{R}$ is equipped with the Gauß–Manin connection $\nabla : \mathcal{R} \to \mathcal{R} \otimes \Omega^1_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$ and the pairing $(\cdot,\cdot)_{\mathcal{R}} : ((-)^*\mathcal{R}) \otimes \mathcal{R} \to \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$ induced from the local system $R^{\vee}_{\mathbb{Z}}$.

3.3.2. The extension across $z = 0$ via de Rham forms

Let ω_1 be the following holomorphic volume form on $Y_1 = \mathrm{Hom}(\boldsymbol{N}, \mathbb{C}^*)$:

$$\omega_1 = \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|} \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n} \quad \text{on each connected component.}$$

This is characterized as a unique translation-invariant holomorphic n-form ω_1 satisfying $\int_{\mathrm{Hom}(\boldsymbol{N},S^1)} \omega_1 = (2\pi\mathbf{i})^n$. By translation, ω_1 defines a holomorphic volume form ω_q on each fiber Y_q. Let $\mathrm{pr} : Y^{\mathrm{o}} \to \mathcal{M}^{\mathrm{o}}$ be the restriction of the family $\mathrm{pr} : Y \to \mathcal{M}$ to $\mathcal{M}^{\mathrm{o}}$. Consider a relative holomorphic n-form φ of $Y^{\mathrm{o}} \times \mathbb{C}^* \to \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ of the form

$$\varphi = f(q,z,y)e^{W_q(y)/z}\omega_q, \quad f(q,z,y) \in \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}\left[y_1^{\pm}, \dots, y_n^{\pm}\right] \tag{52}$$

where $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$ is the analytic structure sheaf. This relative n-form gives a holomorphic section $[\varphi]$ of $\mathcal{R}$ via the integration over Lefschetz thimbles:

$$\langle [\varphi], \Gamma \rangle = \frac{1}{(-2\pi z)^{n/2}} \int_{\Gamma} f(q,z,y)e^{W_q(y)/z}\omega_q \in \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}. \tag{53}$$

The convergence of this integral is ensured by the fact that $f(q,z,y)$ has at most polynomial growth in y and that $\Re(W_q(y)/z)$ goes to $-\infty$ at the end of Γ. More technically, as done in [64], one may prove the convergence of the integral by replacing the end of Γ with a semi-algebraic chain.

Definition 3.13. A section of $\mathcal{R}$ on an open set $U \times \{0 < |z| < \epsilon\} \subset \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ is defined to be *extendible to* $z = 0$ if it is the image of a relative n-form φ of the form (52) such that $f(q,z,y)$ in (52) is regular at $z = 0$. The sections extendible to $z = 0$ define the extension $\mathcal{R}^{(0)}$ of the sheaf $\mathcal{R}$ to $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}$.

Let $\mathcal{R}'$ be the $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$-submodule of $\mathcal{R}$ consisting of the sections which locally arise from relative n-forms φ of the form (52). The Gauß–Manin connection on $\mathcal{R}$ preserves the subsheaf $\mathcal{R}'$. In fact, we have

$$\nabla_a[\varphi] = \left[\left(\partial_a f + \frac{1}{z}(\partial_a W_q) f\right) e^{W_q/z}\omega_q\right],$$
$$\nabla_{z\partial_z}[\varphi] = \left[\left(z\partial_z f - \frac{1}{z} W_q f - \frac{n}{2} f\right) e^{W_q/z}\omega_q\right], \tag{54}$$

where φ is given in (52) and $\partial_a = q_a(\partial/\partial q_a)$. Take a point q in the open subset $\mathcal{M}^{\mathrm{oo}} \subset \mathcal{M}^{\mathrm{o}}$ appearing in Proposition 3.10. Let $\Gamma_1, \dots, \Gamma_N$ be Lefschetz thimbles of $W_q(y)/z$ corresponding to critical points $\mathrm{cr}_1, \dots, \mathrm{cr}_N$. Then we have the following asymptotic expansion as $z \to 0$ with $\arg(z)$ fixed:

$$\frac{1}{(-2\pi z)^{n/2}} \int_{\Gamma_i} f(q,z,y) e^{W_q(y)/z}\omega_q \sim \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|} \frac{e^{W_q(\mathrm{cr}_i)/z}}{\sqrt{\mathrm{Hess}(W_q)(\mathrm{cr}_i)}} \big(f(q,0,\mathrm{cr}_i) + O(z)\big) \tag{55}$$

where $f(q,z,y) \in \mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}[y_1^{\pm}, \dots, y_n^{\pm}]$ is regular at $z=0$ and $\mathrm{Hess}(W_q)$ is the Hessian of W_q calculated in co-ordinates $\log y_1, \dots, \log y_n$. Let $\phi_i(y)$ be a regular function on Y_q which represents the delta-function supported on cr_i in the Jacobi ring $J(W_q)$. Put $\varphi_i = \phi_i(y) e^{W_q/z}\omega_q$. By the asymptotics of $\langle[\varphi_i], \Gamma_j\rangle$, we know that $[\varphi_1], \dots, [\varphi_N]$ form a basis of $\mathcal{R}$ for sufficiently small $|z| > 0$. Since $\mathcal{R}'$ is preserved by the Gauß–Manin connection, we have $\mathcal{R} = \mathcal{R}'$ on the whole $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$. In other words, $\mathcal{R}$ is generated by relative n-forms of the form (52).

Let $\Gamma_1^\vee, \dots, \Gamma_N^\vee$ be the Lefschetz thimbles of $W_q/(-z)$. These are dual to $\Gamma_1, \dots, \Gamma_N$ with respect to the intersection pairing (51). Then the pairing on $\mathcal{R}$ can be written as

$$\big([\varphi(-z)], [\varphi'(z)]\big)_{\mathcal{R}} = \frac{1}{(2\pi\mathbf{i}z)^n} \sum_{i=1}^N \int_{\Gamma_i^\vee} \varphi(-z) \cdot \int_{\Gamma_i} \varphi'(z). \tag{56}$$

When $[\varphi]$ and $[\varphi']$ are extendible to $z=0$, we have from (56) and (55)

$$\big([\varphi], [\varphi']\big)_{\mathcal{R}} \sim \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|^2} \sum_{i=1}^N \frac{f(q,0,\mathrm{cr}_i) f'(q,0,\mathrm{cr}_i)}{\mathrm{Hess}\, W_q(\mathrm{cr}_i)} + O(z)$$

where we put $\varphi = f(q,z,y) e^{W_q(y)/z}\omega_q$ and $\varphi' = f'(q,z,y) e^{W_q(y)/z}\omega_q$. This shows that $([\varphi], [\varphi'])_{\mathcal{R}}$ is regular at $z=0$ and the value at $z=0$ equals the residue pairing on $J(W_q)$. By continuity, we have at all $q \in \mathcal{M}^{\mathrm{o}}$:

$$\big([\varphi], [\varphi']\big)_{\mathcal{R}}\big|_{z=0} = \frac{1}{|\boldsymbol{N}_{\mathrm{tor}}|^2} \mathrm{Res}_{Y^{\mathrm{o}}/\mathcal{M}^{\mathrm{o}}} \left[\frac{f(q,0,y) f'(q,0,y) \frac{dy_1 \cdots dy_n}{y_1 \cdots y_n}}{y_1 \frac{\partial W_q}{\partial y_1}, \dots, y_n \frac{\partial W_q}{\partial y_n}} \right].$$

Let $\phi_1', \dots, \phi_N'$ be an arbitrary basis of the Jacobi ring and put $s_i := [\phi_i'(y) e^{W_q(y)/z}\omega_q]$. Then the Gram matrix $(s_i, s_j)_{\mathcal{R}}$ is non-degenerate in a neighborhood of $z=0$ since the residue pairing is non-degenerate. This implies that $s_1, \dots, s_N$ form a local basis of $\mathcal{R}^{(0)}$ around $z=0$. Summarizing,

Proposition 3.14. *(See [27, Lemma 2.19].) The $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}^*}$-module $\mathcal{R}$ is generated by relative n-forms of the form* (52). *The extension $\mathcal{R}^{(0)}$ of $\mathcal{R}$ to $\mathcal{M}^{\mathrm{o}}\times\mathbb{C}$ is locally free and the pairing on $\mathcal{R}$ extends to a non-degenerate pairing $((-)^*\mathcal{R}^{(0)})\otimes\mathcal{R}^{(0)}\to\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}$.*

In the algebraic construction by Sabbah, the corresponding results were proved in [67, Corollary 10.2] (see also [30, Proposition 2.13]).

The *Euler vector field* E on $\mathcal{M}^{\mathrm{o}}$ is defined by

$$E := \mathrm{pr}_*\left(\sum_{i=1}^m w_i\frac{\partial}{\partial w_i}\right) = \sum_{a=1}^r \rho_a q_a\frac{\partial}{\partial q_a}, \quad \rho_a := \sum_{i=1}^m \mathsf{m}_{ia}. \tag{57}$$

The grading operator Gr acting on sections of $\mathcal{R}^{(0)}$ is defined by

$$\mathsf{Gr}[\varphi] = 2\left[\left(z\frac{\partial f}{\partial z} + \sum_{i=1}^m w_i\frac{\partial f}{\partial w_i}\right)e^{W/z}\omega\right] \tag{58}$$

for a section $[\varphi]$ of the form (52). This grading operator can be written in terms of the Gauß–Manin connection and the Euler vector field (cf. the grading operator (9) for the A-model):

Lemma 3.15. $\mathsf{Gr} = 2(\nabla_E + \nabla_{z\partial_z} + \frac{n}{2})$.

Proof. Using the co-ordinate system (q_a, y_i) on Y in Section 3.2.1, we can write $\sum_{i=1}^m w_i\frac{\partial}{\partial w_i} = E + \sum_{i=1}^n c_i y_i\frac{\partial}{\partial y_i}$ for some $c_i\in\mathbb{Q}$. Here we lift E to a vector field on Y by using the co-ordinates (q_a, y_i). By $(\sum_{i=1}^m w_i\frac{\partial}{\partial w_i})W = W$, we have

$$\begin{aligned}\frac{1}{2}\mathsf{Gr}[\varphi] &= \left[\left(\left(z\partial_z + \sum_{i=1}^m w_i\partial_{w_i}\right)\left(fe^{W/z}\right)\right)\omega\right]\\ &= \left(\nabla_{z\partial_z} + \frac{n}{2} + \nabla_E\right)[\varphi] + \left[\left(\left(\sum_{i=1}^n c_i y_i\partial_{y_i}\right)\left(fe^{W/z}\right)\right)\omega\right].\end{aligned}$$

The second term is zero in cohomology since it is exact. □

Definition 3.16 *(Cf. Definition 2.2).* Let $\pi:\mathcal{M}^{\mathrm{o}}\times\mathbb{C}\to\mathcal{M}^{\mathrm{o}}$ be the projection and $(-):\mathcal{M}^{\mathrm{o}}\times\mathbb{C}\to\mathcal{M}^{\mathrm{o}}\times\mathbb{C}$ be the map sending (q,z) to $(q,-z)$. The *B-model D-module of the LG model* is the tuple $(\mathcal{R}^{(0)},\nabla,(\cdot,\cdot)_{\mathcal{R}^{(0)}})$ consisting of the locally free sheaf $\mathcal{R}^{(0)}$ over $\mathcal{M}^{\mathrm{o}}\times\mathbb{C}$, the meromorphic flat connection (54)

$$\nabla:\mathcal{R}^{(0)}\to\mathcal{R}^{(0)}\left(\mathcal{M}^{\mathrm{o}}\times\{0\}\right)\otimes_{\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}}\left(\pi^*\Omega^1_{\mathcal{M}^{\mathrm{o}}}\oplus\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}\frac{dz}{z}\right)$$

and the ∇-flat pairing (56)

$$(\cdot,\cdot)_{\mathcal{R}^{(0)}}:(-)^*\mathcal{R}^{(0)}\otimes_{\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}}\mathcal{R}^{(0)}\to\mathcal{O}_{\mathcal{M}^{\mathrm{o}}\times\mathbb{C}}$$

satisfying $((-)^*s_1, s_2)_{\mathcal{R}^{(0)}} = (-)^*((-)^*s_2, s_1)_{\mathcal{R}^{(0)}}$. This is also equipped with the grading operator $\mathsf{Gr}: \mathcal{R}^{(0)} \to \mathcal{R}^{(0)}$ in (58).

Note that the B-model D-module is underlain by the integral local system of Lefschetz thimbles.

Proposition 3.17. *The B-model D-module $\mathcal{R}^{(0)}$ is generated by $[e^{W_q/z}\omega_q]$ and its derivatives $z\nabla_{a_1} z\nabla_{a_2}\cdots z\nabla_{a_k}[e^{W_q/z}\omega_q]$ as an $\mathcal{O}_{\mathcal{M}^\circ\times\mathbb{C}}$-module, where $\nabla_a = \nabla_{q_a(\partial/\partial q_a)}$.*

Proof. By the discussion preceding Proposition 3.14, the restriction $\mathcal{R}^{(0)}|_{\mathcal{M}^\circ\times\{0\}}$ is identified with the bundle $J(W)$ of Jacobi rings over $\mathcal{M}^\circ$ by the map $[f(q,z,y)e^{W_q/z}\omega_q] \mapsto [f(q,0,y)]$. Under this identification, the action of $z\nabla_a$ corresponds to the multiplication by $q_a(\partial W_q/\partial q_a)$ by (54). Because $J(W) \cong B(\mathcal{X})$ by Proposition 3.10 and $B(\mathcal{X})$ is generated by p_a's, $J(W)$ is generated by $q_a(\partial W_q/\partial q_a)$ as a $\mathbb{C}[q^{\pm}]$-algebra. Therefore, $\mathcal{R}^{(0)}$ is generated by $z\nabla_{a_1}\cdots z\nabla_{a_k}[e^{W_q/z}\omega_q]$ in the neighborhood of $z=0$. Let $\widetilde{\mathcal{R}}^{(0)}$ be the $\mathcal{O}_{\mathcal{M}^\circ\times\mathbb{C}}$-submodule of $\mathcal{R}^{(0)}$ generated by these derivatives. From $\mathsf{Gr}[e^{W_q/z}\omega_q] = 0$ and Lemma 3.15, one finds that

$$\nabla_{z^2\partial_z}\big[e^{W_q/z}\omega_q\big] = \left(\sum_{a=1}^{r} \rho_a z\nabla_a - \frac{n}{2}z\right)\big[e^{W_q/z}\omega_q\big].$$

Hence $\widetilde{\mathcal{R}}^{(0)}$ is preserved by $\nabla_{z^2\partial_z}$. Therefore, $\widetilde{\mathcal{R}}^{(0)} = \mathcal{R}^{(0)}$. $\square$

4. Mirror symmetry for toric orbifolds and integral structures

Under mirror symmetry, the A-model D-module (quantum D-module) should be isomorphic to the B-model D-module. We give a precise mirror symmetry conjecture for a weak Fano toric orbifold and check that the mirror symmetry matches up the $\widehat{\Gamma}$-integral structure in the A-side and the natural integral structure in the B-side.

4.1. I-function and mirror theorem

A Givental style mirror theorem for a toric orbifold can be stated as the equality of the I-function and the J-function. This has been proved for weak Fano toric manifolds [38] and weighted projective spaces [26]. A general case for toric orbifolds will be proved in [25].

Definition 4.1. (See [25].) The *I-function* of a toric orbifold $\mathcal{X}$ is an $H^*_{\mathrm{orb}}(\mathcal{X})$-valued power series on $\mathcal{M}$ defined by

$$I(q,z) = e^{\sum_{a=1}^{r}\bar{p}_a\log q_a/z} \sum_{d\in\mathbb{K}_{\mathrm{eff}}} q^d \frac{\prod_{i:\langle D_i,d\rangle<0}\prod_{\langle D_i,d\rangle\leqslant\nu<0}(\overline{D}_i + (\langle D_i,d\rangle-\nu)z)}{\prod_{i:\langle D_i,d\rangle>0}\prod_{0\leqslant\nu<\langle D_i,d\rangle}(\overline{D}_i + (\langle D_i,d\rangle-\nu)z)} \mathbf{1}_{v(d)}$$

where $q^d = q_1^{\langle p_1,d\rangle}\ldots q_r^{\langle p_r,d\rangle}$ and the index ν moves in $\mathbb{Z}$. Recall that $\bar{p}_a$ and $\overline{D}_j$ are the images of p_a and D_j under the projection $\mathbb{L}^\vee\otimes\mathbb{Q}\to H^2(\mathcal{X},\mathbb{Q})$. Note that $\bar{p}_a = 0$ for $a > r'$, $\overline{D}_j = 0$ for $j > m'$ and $\langle p_a, d\rangle \geqslant 0$ for $d \in \mathbb{K}_{\mathrm{eff}}$.

Choose $e_0 \in \mathbb{N}$ such that $e_0\mathbb{K} \subset \mathbb{L}$. Then $e^{-\sum_{a=1}^r \overline{p}_a \log q_a/z} I(q,z)$ belongs to $H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}[z,z^{-1}][\![q_1^{1/e_0},\dots,q_r^{1/e_0}]\!]$. The I-function can be also written in the form:

$$I(q,z) = e^{\sum_{a=1}^r \overline{p}_a \log q_a/z} \sum_{d\in\mathbb{K}} q^d \prod_{i=1}^m \frac{\prod_{\nu=\lceil\langle D_i,d\rangle\rceil}^{\infty}(\overline{D}_i + (\langle D_i,d\rangle - \nu)z)}{\prod_{\nu=0}^{\infty}(\overline{D}_i + (\langle D_i,d\rangle - \nu)z)} \mathbf{1}_{v(d)}. \tag{59}$$

Note that all but finite factors cancel in the infinite products. The summand with $d \in \mathbb{K} \setminus \mathbb{K}_{\mathrm{eff}}$ vanishes in $H^*_{\mathrm{orb}}(\mathcal{X})$ because we have $(\prod_{i:\langle D_i,d\rangle\in\mathbb{Z}_{<0}} \overline{D}_i)\mathbf{1}_{v(d)}$ in the numerator and this is zero in $H^*(\mathcal{X}_{v(d)})$ by the presentation (43).

The I-function defines an analytic function when $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$. See Section 3.1.4 for the condition $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$.

Lemma 4.2. *The I-function is a convergent power series in $q_1,\dots,q_r$ if and only if $\hat{\rho}$ is in the closure* $\mathrm{cl}(\widetilde{C}_{\mathcal{X}})$ *of the extended Kähler cone. In this case, the I-function has the asymptotics*

$$I(q,z) = 1 + \frac{\tau(q)}{z} + o\bigl(z^{-1}\bigr)$$

where $\tau(q)$ is a multi-valued function taking values in $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$. The map $q \mapsto \tau(q)$ is called the mirror map.

When $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$, the mirror map τ takes the form

$$\tau(q) = \sum_{a=1}^{r'} (\log q_a)\overline{p}_a + \sum_{j=m'+1}^{m} q^{D_j^\vee}\mathfrak{D}_j + \text{h.o.t.}, \tag{60}$$

where h.o.t. (higher order term) is a power series in $q_1^{1/e_0},\dots,q_r^{1/e_0}$. Thus τ is a local embedding (resp. isomorphism) near $q=0$ if $\overline{p}_1,\dots,\overline{p}_{r'},\mathfrak{D}_{m'+1},\dots,\mathfrak{D}_m$ are linearly independent (resp. basis of $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$). See (45) for $\mathfrak{D}_j$. The following "mirror theorem" will be proved in [25].

Conjecture 4.3. *Assume that $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$. Then the I-function and the J-function coincide via the co-ordinate change $\tau = \tau(q)$:*

$$I(q,z) = J\bigl(\tau(q),z\bigr),$$

where $\tau(q)$ is the mirror map in Lemma 4.2.

We remark that the equality $I = J$ above is consistent with monodromy transformations on $\mathcal{M}$. Take a loop $[0,1] \ni \theta \mapsto e^{-2\pi\mathbf{i}\xi\theta}q = (e^{-2\pi\mathbf{i}\xi_1\theta}q_1,\dots,e^{-2\pi\mathbf{i}\xi_r\theta}q_r) \in \mathcal{M}$ for $\xi = \sum_{a=1}^r \xi_a p_a \in \mathbb{L}^\vee$. The monodromy of $I(q,z)$ along this loop is given by

$$I\bigl(e^{-2\pi\mathbf{i}\xi}q,z\bigr) = G^{\mathcal{H}}(\xi)I(q,z)$$

where $G^{\mathcal{H}}(\xi) = G^{\mathcal{H}}([\xi])$ is the Galois action (29) of the class $[\xi]$ in $\mathbb{L}^\vee / \sum_{j>m'} \mathbb{Z} D_j \cong H^2(\mathcal{X}, \mathbb{Z})$. Therefore, we have

$$\tau\left(e^{-2\pi \mathbf{i} \xi} q\right) = G(\xi)\tau(q) \tag{61}$$

where $G(\xi) = G([\xi])$ is given in (10). These two equations are compatible with the behavior (32) of $J(\tau, z)$. This moreover shows that τ induces a single-valued map

$$\tau : \left\{(q_1, \dots, q_r) \in \mathcal{M};\ 0 < |q_a| < \epsilon\right\} \longrightarrow H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X}) / H^2(\mathcal{X}, \mathbb{Z}) \tag{62}$$

for a sufficiently small $\epsilon > 0$.

4.2. GKZ-system and an isomorphism of D-modules

The mirror theorem $I = J$ implies that the B-model D-module is isomorphic to the A-model D-module (quantum D-module) pulled back by the mirror map τ. The I-function generates a confluent version of the Gelfand–Kapranov–Zelevinsky (GKZ) D-module [36] studied by Adolphson [4]. This turns out to be isomorphic to the B-model D-module.

Set $\partial_a := q_a(\partial/\partial q_a)$. We write $q^\pm, z\partial$ as shorthand for $q_1^\pm, \dots, q_r^\pm$ and $z\partial_1, \dots, z\partial_r$. Introduce a differential operator $\mathcal{P}_d \in \mathbb{C}[z, q^\pm]\langle z\partial\rangle$ for $d \in \mathbb{L}$ as

$$\mathcal{P}_d := q^d \prod_{i:\ \langle D_i, d\rangle < 0} \prod_{\nu=0}^{-\langle D_i, d\rangle - 1} (\mathcal{D}_i - \nu z) - \prod_{i:\ \langle D_i, d\rangle > 0} \prod_{\nu=0}^{\langle D_i, d\rangle - 1} (\mathcal{D}_i - \nu z).$$

Here we put $\mathcal{D}_i := \sum_{a=1}^r \mathsf{m}_{ia} z\partial_a$. Note that $\mathcal{P}_d$ is well-defined since $\langle D_i, d\rangle \in \mathbb{Z}$ when $d \in \mathbb{L}$. Define the GKZ D-module M_{GKZ} by

$$M_{\mathrm{GKZ}} := \mathbb{C}\left[z, q^\pm\right]\langle z\partial\rangle \Big/ \sum_{d \in \mathbb{L}} \mathbb{C}\left[z, q^\pm\right]\langle z\partial\rangle \mathcal{P}_d.$$

A grading operator Gr on M_{GKZ} is defined by

$$\mathsf{Gr}\left(\left[f(z, q)(z\partial)^k\right]\right) = \left[\left(2|k| f + 2z\frac{\partial f}{\partial z} + 2Ef\right)(z\partial)^k\right], \tag{63}$$

where $k \in (\mathbb{Z}_{\geqslant 0})^r$ is a multi-index, $|k| = \sum_{a=1}^r k_a$ and $E = \sum_{a=1}^r \rho_a \partial_a$ is the Euler vector field (57) of the B-model D-module. This is well-defined because of the homogeneity of the relation $\mathcal{P}_d$. Using the grading operator Gr, we can introduce a flat connection $\nabla : M_{\mathrm{GKZ}} \to \frac{1}{z} M_{\mathrm{GKZ}} \otimes (\mathbb{C}\frac{dz}{z} \oplus \bigoplus_{a=1}^r \mathbb{C}\frac{dq_a}{q_a})$ by (cf. (9), Lemma 3.15)

$$\nabla_a\left[P(z, q, z\partial)\right] := \frac{1}{z}\left[z\partial_a P(z, q, z\partial)\right], \quad 1 \leqslant a \leqslant r;$$

$$\nabla_{z\partial_z} := \frac{1}{2}\mathsf{Gr} - \nabla_E - \frac{n}{2}.$$

Proposition 4.4. *The $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}}[z]$-module $\widetilde{M}_{\mathrm{GKZ}} := M_{\mathrm{GKZ}} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{\mathcal{M}^{\mathrm{o}}}[z]$ is finitely generated as an $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}}[z]$-module. The fiber of $\widetilde{M}_{\mathrm{GKZ}}$ at every point $(q,z) \in \mathcal{M}^{\mathrm{o}} \times \mathbb{C}$ has dimension less than or equal to $|\boldsymbol{N}_{\mathrm{tor}}| \times n!\,\mathrm{Vol}(\widehat{S})$.*

Proof. For a differential operator $P = \sum_k P_k(z,q)(z\partial)^k \in \mathcal{O}_{\mathcal{M}^{\mathrm{o}}}[z]\langle z\partial\rangle$ of rank s, its principal symbol $\sigma(P)$ is defined to be $\sigma(P) := \sum_{|k|=s} P_k(z,q)\mathsf{p}^k$ (the highest order term in $z\partial$), where $k \in (\mathbb{Z}_{\geqslant 0})^r$ is a multi-index and $|k| = \sum_{a=1}^r k_a$. For example,

$$\sigma(\mathcal{P}_d) = \begin{cases} -\prod_{i:\ \langle D_i,d\rangle>0} \mathsf{w}_i^{\langle D_i,d\rangle} & \text{if } \langle\hat{\rho},d\rangle > 0; \\ q^d \prod_{i:\ \langle D_i,d\rangle<0} \mathsf{w}_i^{-\langle D_i,d\rangle} - \prod_{i:\ \langle D_i,d\rangle>0} \mathsf{w}_i^{\langle D_i,d\rangle} & \text{if } \langle\hat{\rho},d\rangle = 0; \\ q^d \prod_{i:\ \langle D_i,d\rangle<0} \mathsf{w}_i^{-\langle D_i,d\rangle} & \text{if } \langle\hat{\rho},d\rangle < 0. \end{cases}$$

Recall that $\mathsf{w}_i = \sum_{a=1}^r \mathsf{m}_{ia}\mathsf{p}_a$ and $\hat{\rho} = \sum_{i=1}^m D_i \in \mathbb{L}^\vee$. By a standard argument, we know that $\widetilde{M}_{\mathrm{GKZ}}$ is finitely generated as an $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}}[z]$-module once we know that

$$B_{\mathrm{c}}(\mathcal{X}) := \mathcal{O}_{\mathcal{M}^{\mathrm{o}}}[\mathsf{p}_1,\dots,\mathsf{p}_r]/\big\langle\sigma(\mathcal{P}_d);\ d \in \mathbb{L}\big\rangle$$

is a finitely generated $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}}$-module. Adolphson [4, Section 3] showed that the characteristic variety of the GKZ D-module is supported on the zero section when the corresponding Laurent polynomials W_q are non-degenerate. Although the D-module in [4] is a little different from ours and it is assumed that $\boldsymbol{N}$ is torsion free there, the same argument as in [4, Section 3] shows[5] that if $\sigma(\mathcal{P}_d)(q,\mathsf{p}) = 0$ for all $d \in \mathbb{L}$,

- either $(\mathsf{p}_1,\dots,\mathsf{p}_r) = 0$ or there exists a proper face Δ of $\widehat{S}$ such that $\mathsf{w}_i \neq 0$ if and only if $b_i \in \Delta$ [4, Lemmas 3.1, 3.2];
- in the latter case, $W_{q,\Delta}(y)$ has a critical point in $(\mathbb{C}^*)^n$ [4, Lemma 3.3].

Thus, $\mathsf{p}_1 = \cdots = \mathsf{p}_r = 0$ if $q \in \mathcal{M}^{\mathrm{o}}$ and $\sigma(\mathcal{P}_d)(q,\mathsf{p}) = 0$ for all $d \in \mathbb{L}$. By Hilbert's Nullstellensats, p_a^k vanishes in $B_{\mathrm{c}}(\mathcal{X})$ for a sufficiently big $k > 0$, so $B_{\mathrm{c}}(\mathcal{X})$ is finitely generated as an $\mathcal{O}_{\mathcal{M}^{\mathrm{o}}}$-module.

Since a coherent sheaf admitting a flat connection is locally free, we know that $\widetilde{M}_{\mathrm{GKZ}}$ is locally free away from $z = 0$. On the other hand, the restriction to $z = 0$ of $\widetilde{M}_{\mathrm{GKZ}}$ is isomorphic to the Batyrev ring:

$$\widetilde{M}_{\mathrm{GKZ}}/z\widetilde{M}_{\mathrm{GKZ}} \cong B(\mathcal{X}) \otimes_{\mathbb{C}[q^{\pm}]} \mathcal{O}_{\mathcal{M}^{\mathrm{o}}}.$$

This is isomorphic to the Jacobi ring by Proposition 3.10(i) and of rank $|\boldsymbol{N}_{\mathrm{tor}}| \times n!\,\mathrm{Vol}(\widehat{S})$ by Proposition 3.7(iii). The conclusion follows from Nakayama's lemma. □

Remark 4.5. The rank of the "confluent" GKZ D-module was calculated in [4] under weaker assumptions (it is not assumed that $\widehat{S}$ contains the origin in its interior). Our D-module M_{GKZ} is a dimensional reduction of the original GKZ-system in [4,36] and is also referred to as the *Horn system*. It is also homogenized by z. The argument above is an adaptation (and a shortcut) of [4]

[5] Note that $\sigma(P_d)$ and w_i correspond to $\sigma(\square_l)$ and y_i in [4].

to our D-module M_{GKZ}. We will see in the proof of Proposition 4.8 that $\widetilde{M}_{\mathrm{GKZ}}$ is exactly of rank $|\boldsymbol{N}_{\mathrm{tor}}| \times n!\,\mathrm{Vol}(\widehat{S})$.

Lemma 4.6. *Assume that $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$. Then the I-function and the oscillatory integrals (associated to the LG model in Section 3.2) satisfy the GKZ-type differential equations:*

$$\mathcal{P}_d I(q,z) = \mathcal{P}_d\left(\int_\Gamma e^{W_q/z}\omega_q\right) = 0, \quad d \in \mathbb{L},$$

where Γ is an arbitrary Lefschetz thimble.

Proof. We use the expression (59) of the I-function. Put

$$\Box_d := \prod_{i=1}^m \frac{\prod_{\nu=\lceil\langle D_i,d\rangle\rceil}^\infty (\overline{D}_i + (\langle D_i,d\rangle - \nu)z)}{\prod_{\nu=0}^\infty (\overline{D}_i + (\langle D_i,d\rangle - \nu)z)}, \quad d \in \mathbb{L}\otimes\mathbb{Q}.$$

Using $\mathcal{D}_i(e^{\sum_{a=1}^r \bar{p}_a \log q_a/z} q^\delta) = e^{\sum_{a=1}^r \bar{p}_a \log q_a/z} q^\delta(\overline{D}_i + \langle D_i,\delta\rangle z)$, one finds that $\mathcal{P}_d I(q,z) = 0$ for $d \in \mathbb{L}$ is equivalent to the difference equation:

$$\Box_{\delta-d} \prod_{i:\ \langle D_i,d\rangle<0} \prod_{\nu=0}^{-\langle D_i,d\rangle-1} \big(\overline{D}_i + \big(\langle D_i,\delta\rangle - \nu\big)z\big) = \Box_\delta \prod_{i:\ \langle D_i,d\rangle>0} \prod_{\nu=0}^{\langle D_i,d\rangle-1} \big(\overline{D}_i + \big(\langle D_i,\delta\rangle - \nu\big)z\big)$$

for all $\delta \in \mathbb{K}$. This is easy to check.

We omit the proof for oscillatory integrals since it is completely parallel to the case of toric manifolds (see e.g. [47, Proposition 5.1]). □

Lemma 4.7. *For $\delta \in \mathbb{K}$ such that $\langle D_i,\delta\rangle > 0$ for all i, we have*

$$q^{-\delta}\left(\prod_{i=1}^m \prod_{\nu=0}^{\lceil\langle D_i,\delta\rangle\rceil-1} (\mathcal{D}_i - \nu z)\right) I(q,z) = e^{\sum_{a=1}^r \bar{p}_a \log q_a/z}\big(\mathbf{1}_{v(\delta)} + O\big(q^{1/e_0}\big)\big)$$

for $e_0 \in \mathbb{N}$ satisfying $e_0\mathbb{K} \subset \mathbb{L}$.

Proof. Using the expression (59), we find that the left-hand side is

$$e^{\sum_{a=1}^r \bar{p}_a \log q_a/z} \sum_{d\in\mathbb{K}} q^{d-\delta} \prod_{i=1}^m \frac{\prod_{\nu=\lceil\langle D_i,d\rangle\rceil}^\infty (\overline{D}_i + (\langle D_i,d\rangle - \nu)z)}{\prod_{\nu=\lceil\langle D_i,\delta\rangle\rceil}^\infty (\overline{D}_i + (\langle D_i,d\rangle - \nu)z)} \mathbf{1}_{v(d)}.$$

We claim that the summand vanishes when $\langle p_a, d-\delta\rangle < 0$ for some a. Note that there remains a factor $(\prod_{i:\langle D_i,d\rangle\in\mathbb{Z},\langle D_i,d\rangle<\langle D_i,\delta\rangle} \overline{D}_i)\mathbf{1}_{v(d)}$ in the numerator. Thus by (43), it suffices to show that $I := \{i;\ \langle D_i,d\rangle \in \mathbb{Z},\ \langle D_i,d-\delta\rangle \geqslant 0\} \notin \mathcal{A}$. Suppose $I \in \mathcal{A}$. Because $p_a \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$, there exists $c_i \geqslant 0$ for $i \in I$ such that $p_a = \sum_{i\in I} c_i D_i$ by the definition of $\widetilde{C}_{\mathcal{X}}$. Then $\langle p_a, d-\delta\rangle = \sum_{i\in I} c_i\langle D_i, d-\delta\rangle \geqslant 0$. This is a contradiction. □

By the condition (C) in Section 3.1.1, for each $v \in \mathrm{Box}$, there exists $\delta \in \mathbb{K}$ such that $v(\delta) = v$ and $\langle D_i, \delta\rangle > 0$ for all i. Thus by Lemma 4.7 and the presentation (42), (43) of $H^*_{\mathrm{orb}}(\mathcal{X})$, we can find differential operators $P_i(z, q, z\partial) \in \mathbb{C}[z, q^{\pm 1/e_0}]\langle z\partial\rangle$, $1 \leqslant i \leqslant N$ such that

$$P_i(z, q, z\partial) I(q, z) = e^{\sum_{a=1}^r \bar{p}_a \log q_a/z}\big(\phi_i + O\big(q^{1/e_0}\big)\big), \tag{64}$$

where ϕ_i, $1 \leqslant i \leqslant N$ is a basis of $H^*_{\mathrm{orb}}(\mathcal{X})$. Under Conjecture 4.3, we have

$$P_i(z, q, z\partial) I(q, z) = P_i(z, q, z\partial) J\big(\tau(q), z\big) = L\big(\tau(q), z\big)^{-1} P_i(z, q, z\tau^*\nabla)\mathbf{1}. \tag{65}$$

Here $L(\tau, z)$ is the fundamental solution in (11) and ∇ is the Dubrovin connection: $\tau^*\nabla$ is shorthand for $\tau^*\nabla_1, \dots, \tau^*\nabla_r$ with $\tau^*\nabla_a := \nabla_{\tau_*(q_a(\partial/\partial q_a))}$. Since $L(\tau(q), z)^{-1} = \mathbf{1} + O(z^{-1})$ (regular at $z = \infty$) and $P_i(z, q, z\tau^*\nabla)\mathbf{1}$ is regular at $z = 0$, Eq. (65) can be viewed as the Birkhoff factorization (see e.g. [65]) of the element

$$S^1 \ni z \longmapsto \begin{bmatrix} | & & | \\ P_1 I & \dots & P_N I \\ | & & | \end{bmatrix} \sim e^{\sum_{a=1}^r \bar{p}_a \log q_a/z}\big(\mathbf{1} + O\big(q^{1/e_0}\big)\big)$$

in the loop group $LGL(N, \mathbb{C})$. Here the asymptotics (64) show that the matrix $[P_1 I, \dots, P_N I]$ is invertible and admits the (unique) Birkhoff factorization[6] when $|q_a|$ is sufficiently small. In particular, it follows that the fundamental solution $L(\tau(q), z)$ is analytic for small values of $|q_a|$ and that the quantum cohomology/D-module is convergent over the image of τ. Note that by (64), we have

$$P_i(z, q, z\tau^*\nabla)\mathbf{1} = \phi_i + O\big(q^{1/e_0}\big) \tag{66}$$

and that these vectors form a basis of $H^*_{\mathrm{orb}}(\mathcal{X})$ for small $|q_a|$.

Now we formulate toric mirror symmetry as an isomorphism of D-modules.

Proposition 4.8. *Assume that our initial data satisfies $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$ and that Conjecture* 4.3 *holds for $\mathcal{X}$. The B-model D-module (in Definition* 3.16*) is isomorphic to the pull back of the A-model D-module (in Definition* 2.2*) under the mirror map τ in* (62)*:*

$$\mathrm{Mir}\colon \big(\mathcal{R}^{(0)}, \nabla, (\cdot,\cdot)_{\mathcal{R}^{(0)}}\big)\big|_{V_\epsilon \times \mathbb{C}} \cong (\tau \times \mathrm{id})^*\big(\big(F, \nabla, (\cdot,\cdot)_F\big)/H^2(\mathcal{X}, \mathbb{Z})\big)$$

where $V_\epsilon = \{(q_1, \dots, q_r) \in \mathcal{M};\ 0 < |q_a| < \epsilon\}$ and $\epsilon > 0$ is a sufficiently small real number. The right-hand side is the quotient by the Galois action. The isomorphism Mir *sends $[e^{W_q/z}\omega_q]$ to the unit section $\mathbf{1}$ of F.*

Proof. First we identify the GKZ D-module with the A-model D-module. Consider a D-module homomorphism:

[6] The convergence of quantum cohomology is not a priori known. However the Birkhoff factorization here can be done uniquely over the ring of formal power series in $q_1^{1/e_0}, \dots, q_r^{1/e_0}$ after removing the factor $e^{\sum_{a=1}^r \bar{p}_a \log q_a/z}$. See [46, Theorem 3.9].

$$
\begin{aligned}
M_{\mathrm{GKZ}} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{V_\epsilon \times \mathbb{C}} &\longrightarrow \mathcal{O}\big((\tau \times \mathrm{id})^*\big(F/H^2(\mathcal{X},\mathbb{Z})\big)\big), \\
\big[P(z,q,z\partial)\big] &\longmapsto P(z,q,z\tau^*\nabla)\mathbf{1}.
\end{aligned} \tag{67}
$$

We claim that this map is an isomorphism. By Lemma 4.6 and (65), this map is well-defined. Eq. (66) shows that this is surjective for some small $\epsilon > 0$. By Lemma 3.8, we may assume $V_\epsilon \subset \mathcal{M}^{\mathrm{o}}$. Then we can deduce the claim by comparing the ranks (Proposition 4.4 and Lemma 3.9). Next consider a D-module homomorphism:

$$
\begin{aligned}
M_{\mathrm{GKZ}} \otimes_{\mathbb{C}[z,q^{\pm}]} \mathcal{O}_{V_\epsilon \times \mathbb{C}} &\longrightarrow \mathcal{R}^{(0)}\big|_{V_\epsilon \times \mathbb{C}}, \\
\big[P(z,q,z\partial)\big] &\longmapsto P(z,q,z\nabla)\big[e^{W_q/z}\omega_q\big],
\end{aligned} \tag{68}
$$

where ∇ is the flat connection of the B-model D-module. This is well-defined by Lemma 4.6 and surjective by Proposition 3.17. Thus it is an isomorphism again by comparison of the ranks (Propositions 3.12 and 4.4). By composing the two isomorphisms (67), (68), we get the desired isomorphism $\mathrm{Mir}: \mathcal{R}^{(0)}|_{V_\epsilon \times \mathbb{C}} \cong \mathcal{O}((\tau \times \mathrm{id})^*(F/H^2(\mathcal{X},\mathbb{Z})))$ sending $[e^{W_q/z}\omega_q]$ to $\mathbf{1}$.

It is clear that $\nabla_a = \nabla_{q^a(\partial/\partial q_a)}$ corresponds to $\tau^*\nabla_a$ under the map Mir. It is easy to check that the isomorphisms (67) and (68) preserve the grading operators (see (9), (63) and (58); we use the homogeneity of the series $e^{-\sum_{a=1}^{r} \bar{p}_a \log q_a/z} I(q,z)$). Hence Mir preserves Gr and so sends $\nabla_{z\partial_z}$ to $\tau^*\nabla_{z\partial_z}$ (we use the fact that τ preserves the Euler vector field).

The proof of $(\cdot,\cdot)_{\mathcal{R}^{(0)}} = (\tau \times \mathrm{id})^*(\cdot,\cdot)_F$ is given in Appendix A.3. $\square$

Corollary 4.9. *Under the same assumptions as Proposition* 4.8*, the quantum cohomology of a toric orbifold $\mathcal{X}$ is generically semisimple, i.e. $(H^*_{\mathrm{orb}}(\mathcal{X}), \circ_\tau)$ is isomorphic to the direct sum of $\mathbb{C}$ as a ring for a generic $\tau \in U$.*

Proof. The quantum cohomology of $\mathcal{X}$ is identified with the Jacobi ring $J(W_q)$ of the mirror. The conclusion follows from Proposition 3.10(ii). $\square$

Remark 4.10. When $\mathcal{X}$ is not weak Fano, the mirror theorem Conjecture 4.3 should be replaced with the Coates and Givental [22] style statement that the I-function is on the Givental's Lagrangian cone (30). The D-module isomorphism cannot hold since the ranks are different ($|N_{\mathrm{tor}}| \times n!\,\mathrm{Vol}(\widehat{S}) > \dim H_{\mathrm{orb}}(\mathcal{X})$), but the quantum D-module should be isomorphic to a certain completion of the GKZ D-module at the large radius limit $q = 0$ and the semisimplicity of quantum cohomology should still hold. The details will appear in [25]. (See [47,48] for toric manifolds.)

4.3. The integral structures match

Theorem 4.11. *Let $\mathcal{X}$ be a weak Fano projective toric orbifold defined by initial data satisfying $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$. Assume that Conjecture* 4.3 *and Assumption* 2.7(c) *hold for $\mathcal{X}$. Then the mirror isomorphism* Mir *in Proposition* 4.8 *sends the natural integral structure (lattice of Lefschetz thimbles) of the B-model D-module to the $\widehat{\Gamma}$-integral structure (Definition* 2.9*) of the A-model D-module.*

First we draw a corollary on Dubrovin's conjecture [32, 4.2.2] from this theorem. Since the $\widehat{\Gamma}$-integral structure is defined to be the image of the K-group, we can identify the integral lattice $R^\vee_{\mathbb{Z},(q,z)}$ generated by Lefschetz thimbles with (the dual[7] of) the K-group $K(\mathcal{X})$. This also identifies the pairings on the both sides. Let $V_1,\dots,V_N \in K(\mathcal{X})$ correspond to a basis $\Gamma_1,\dots,\Gamma_N$ of Lefschetz thimbles whose images under W_q are straight half-lines. Then we have

$$\chi\big(V_i^\vee \otimes V_j\big) = \sharp\big(\Gamma_i \cap e^{\pi \mathbf{i}}\Gamma_j\big),$$

where $e^{\pi\mathbf{i}}\Gamma_j$ is the parallel translate of $\Gamma_j \in H_n(Y_q, \{y;\ \Re(W_q(y)/z) \ll 0\})$ along the path $[0,1] \ni \theta \mapsto e^{\pi\mathbf{i}\theta} z$ (cf. (17)). On the other hand, the quantum differential equation in z

$$\nabla_{z\partial_z}\psi(z) = \left(z\frac{\partial}{\partial z} - \frac{1}{z}E\circ + \mu\right)\psi(z) = 0 \tag{69}$$

is irregular singular at $z=0$ and defines a *Stokes matrix* (see [32,33]). Under mirror symmetry, the Stokes matrix is given by the intersection numbers $\sharp(\Gamma_i \cap e^{\pi\mathbf{i}}\Gamma_j)$ by Picard–Lefschetz theory (since a solution ψ is given by oscillatory integrals over Γ_i's; see e.g. [18,73]). Hence,

Corollary 4.12 *(K-group version of Dubrovin's conjecture). Under the same assumptions as Theorem* 4.11*, there exist $V_1,\dots,V_N \in K(\mathcal{X})$ such that the matrix $S=(S_{ij})$, $S_{ij} := \chi(V_i^\vee \otimes V_j)$ is a Stokes matrix of the quantum differential equation of $\mathcal{X}$. (In particular, S is upper-triangular and $S_{ii}=1$.)*

Remark 4.13. Dubrovin's conjecture [32] furthermore asserts that $V_1,\dots,V_N$ here should come from an *exceptional collection* in the derived category. This should follow from homological mirror symmetry. For toric varieties, different versions of homological mirror symmetry have been obtained (or announced) by Abouzaid [1], Fang, Liu, Treumann, and Zaslow [34] and Bondal and Ruan [9]. The author is not sure if their results imply Dubrovin's conjecture since, except for the approach by Bondal–Ruan, they do not deal with Lefschetz thimbles directly. For a weighted projective space $\mathcal{X}$, $\Gamma_1,\dots,\Gamma_N$ are the monodromy transforms (in q) of the real Lefschetz thimble $\Gamma_{\mathbb{R}}$ (see Theorem 4.14 below), so these actually correspond to an exceptional collection $\mathcal{O}(-a), \mathcal{O}(-a+1),\dots,\mathcal{O}(b)$ for some a,b. (Dubrovin's conjecture for $\mathcal{X}=\mathbb{P}^n$ was proved by Guzzetti [40].) For general $\mathcal{X}$, it might be difficult to calculate V_i corresponding to Γ_i whose image under W_q is a straight half-line.

Theorem 4.11 follows from the matching of the central charges from quantum cohomology and LG model. Consider the fibration formed by real points on (46):

$$\mathbf{1} \longrightarrow \mathrm{Hom}(\boldsymbol{N},\mathbb{R}_{>0}) \longrightarrow Y_{\mathbb{R}} := (\mathbb{R}_{>0})^m \xrightarrow{\mathrm{pr}|_{Y_{\mathbb{R}}}} \mathcal{M}_{\mathbb{R}} := \mathrm{Hom}(\mathbb{L},\mathbb{R}_{>0}) \longrightarrow \mathbf{1}.$$

Here we regard $\mathbb{R}_{>0}$ as an abelian group with respect to the multiplication. This exact sequence splits and the section given by the matrix (ℓ_{ia}) in Section 3.2.1 is single-valued over the real locus $\mathcal{M}_{\mathbb{R}}$. For $q \in \mathcal{M}_{\mathbb{R}}$, the real Lefschetz thimble $\Gamma_{\mathbb{R}} \subset Y_q$ is defined to be

$$\Gamma_{\mathbb{R}} := Y_q \cap Y_{\mathbb{R}} = \big\{(y_1,\dots,y_n) \in Y_q;\ y_i > 0\big\} \cong \mathrm{Hom}(\boldsymbol{N},\mathbb{R}_{>0}).$$

[7] We identify the dual of the K-group with the K-group itself by the Mukai pairing.

The oscillatory integral $\int_{\Gamma_{\mathbb{R}}} e^{-W_q/z}\omega_q$ is well-defined for $q \in \mathcal{M}_{\mathbb{R}}$ and $z > 0$. We also define $\Gamma_{\mathrm{c}} \subset Y_q$ to be the parallel translate of the monodromy-invariant compact cycle

$$\Gamma_{\mathrm{c}} := \mathrm{Hom}\big(\boldsymbol{N}, S^1\big) \subset Y_{q=1}.$$

Note that Γ_{c} is a disjoint union of $|\boldsymbol{N}_{\mathrm{tor}}|$ number of tori $(S^1)^n$.

Theorem 4.14. *Assume that $\hat{\rho} \in \mathrm{cl}(\widetilde{C}_{\mathcal{X}})$ and that Conjecture* 4.3 *holds. The quantum cohomology central charges* (25) *of the structure sheaf $\mathcal{O}_{\mathcal{X}}$ and the skyscraper sheaf $\mathcal{O}_{\mathrm{pt}}$ are given by the oscillatory integrals over the real Lefschetz thimble $\Gamma_{\mathbb{R}}$ and the compact cycle Γ_{c} respectively:*

$$Z(\mathcal{O}_{\mathcal{X}})\big(\tau(q), z\big) = \frac{1}{(2\pi \mathbf{i})^n} \int\limits_{\Gamma_{\mathbb{R}} \subset Y_q} e^{-W_q/z}\omega_q, \quad q \in \mathcal{M}_{\mathbb{R}}, z > 0; \tag{70}$$

$$Z(\mathcal{O}_{\mathrm{pt}})\big(\tau(q), z\big) = \frac{1}{(2\pi \mathbf{i})^n} \int\limits_{\Gamma_{\mathrm{c}} \subset Y_q} e^{-W_q/z}\omega_q, \quad (q, z) \in \mathcal{M} \times \mathbb{C}^*, \tag{71}$$

where $\tau(q)$ is the mirror map. In Eq. (70)*, the branches of* $\log z$*, $\tau(q)$ in the definition of the left-hand side is chosen so that* $\log z \in \mathbb{R}$, $\tau(q) \in H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X}, \mathbb{R})$.

The right-hand sides of (70), (71) are considered as the *LG central charges* (called *BPS mass* in [43]) of $\Gamma_{\mathbb{R}}$ and Γ_{c}. This theorem corresponds to a compact toric version of Hosono's conjecture [45, Conjecture 2.2], which was stated for Calabi–Yau complete intersections in terms of hypergeometric series (in place of $Z(V)$) and periods (in place of oscillatory integrals).

Remark 4.15. (i) The equality (70) of central charges solves a connection problem for the quantum differential equation (69) in z which is regular singular at $z = \infty$ and irregular singular at $z = 0$. The oscillatory integral admits an asymptotic expansion at $z = 0$ and $Z(\mathcal{O}_{\mathcal{X}})$ is (by definition) expanded in a power series in z^{-1}.

(ii) This theorem suggests that, under homological mirror symmetry, the thimble $\Gamma_{\mathbb{R}}$ (or Γ_{c}) (an object of Fukaya–Seidel category of the LG model), should correspond to the structure sheaf $\mathcal{O}_{\mathcal{X}}$ (or $\mathcal{O}_{\mathrm{pt}}$) (an object of the derived category of coherent sheaves on $\mathcal{X}$). This correspondence is consistent with the Strominger–Yau–Zaslow (SYZ) picture [69]. The cycle $\Gamma_{\mathbb{R}}$ (resp. Γ_{c}) gives a Lagrangian section (resp. fiber) of the SYZ fibration, so should correspond to the structure (resp. skyscraper) sheaf.

4.3.1. Proof of Theorem 4.11 under Theorem 4.14

Fix a point $q \in \mathcal{M}_{\mathbb{R}}$ and $z > 0$. The mirror isomorphism Mir in Proposition 4.8 defines a map

$$R^{\vee}_{(q,-z)} = H_n\big(Y_q, \big\{y \in Y_q;\ \Re\big(W_q(y)/(-z)\big) \ll 0\big\}\big) \to \mathcal{S}(\mathcal{X}), \quad \Gamma \mapsto s_{\Gamma}(\tau, z),$$

such that

$$\big(\mathrm{Mir}[\varphi], s_{\Gamma}\big(\tau(q), z\big)\big)_{\mathrm{orb}} = \big\langle [\varphi], \Gamma \big\rangle, \quad \forall [\varphi] \in \mathcal{R}^{(0)}_{(q,-z)},$$

where the right-hand side is the pairing in (53) and $\log z$ and $\tau(q)$ in the left-hand side are taken to be real as above. Let $\widetilde{\mathcal{S}}(\mathcal{X})_{\mathbb{Z}}$ be the image of this map. We need to show that $\widetilde{\mathcal{S}}(\mathcal{X})_{\mathbb{Z}}$ coincides with the $\widehat{\Gamma}$-integral structure $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$. From the definition (25) of $Z(\mathcal{O}_{\mathcal{X}})$, one can rewrite (70) as

$$\big(\mathbf{1}, \mathcal{Z}_K(\mathcal{O}_{\mathcal{X}})\big(\tau(q), z\big)\big)_{\mathrm{orb}} = \big\langle\big[e^{-W_q/z}\omega_q\big], \Gamma_{\mathbb{R}}\big\rangle.$$

Because Mir sends $[e^{-W_q/z}\omega_q] \in \mathcal{R}^{(0)}_{(q,-z)}$ to $\mathbf{1} \in F_{(\tau(q),-z)}$ and the B-model D-module is generated by $[e^{-W_q/z}\omega_q]$ and its derivatives (Proposition 3.17), we have $\mathcal{Z}_K(\mathcal{O}_{\mathcal{X}}) = s_{\Gamma_{\mathbb{R}}} \in \widetilde{\mathcal{S}}(\mathcal{X})_{\mathbb{Z}}$. Because $K(\mathcal{X})$ is generated by line bundles [10] and $\widetilde{\mathcal{S}}(\mathcal{X})_{\mathbb{Z}}$ is preserved by the Galois action, we have $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} = \mathcal{Z}_K(\mathbb{Z}[\mathrm{Pic}(\mathcal{X})]\mathcal{O}_{\mathcal{X}}) \subset \widetilde{\mathcal{S}}(\mathcal{X})_{\mathbb{Z}}$. Because the pairing of the A-model and B-model coincide, $\widetilde{\mathcal{S}}(\mathcal{X})_{\mathbb{Z}}$ is a unimodular lattice in $\mathcal{S}(\mathcal{X})$. Under Assumption 2.7(c), $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ is also unimodular. Therefore $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} = \widetilde{\mathcal{S}}(\mathcal{X})_{\mathbb{Z}}$.

4.4. Equivariant perturbation

Here we prove Theorem 4.14. We will make use of Givental's *equivariant mirror* which gives a perturbation of oscillatory integrals. This is considered as a mirror of equivariant quantum cohomology of toric orbifolds. We prove an equivariant version of (70) and conclude (70) by taking the non-equivariant limit. In this article, we do not formulate equivariant mirror symmetry.

4.4.1. Equivariant oscillatory integrals

Let $T := (\mathbb{C}^*)^m$ act on our toric orbifold $\mathcal{X} = \mathbb{C}^m/\!/\mathbb{T}$ via the diagonal action of $(\mathbb{C}^*)^m$ on $\mathbb{C}^m$. Let $-\lambda_1, \dots, -\lambda_m$ be the equivariant variables corresponding to generators of $H^*_T(\mathrm{pt})$. Here λ_i denotes either a cohomology class or a complex number depending on the context. Givental's equivariant mirror [38] is given by the following perturbed potential W^λ:

$$W^\lambda := \sum_{i=1}^m (w_i + \lambda_i \log w_i) = W + \sum_{i=1}^m \lambda_i \log w_i.$$

Hereafter λ_i denotes a complex number. This is a multi-valued function on each fiber Y_q. Morse theory for $\Re(W^\lambda(y)/z)$ will compute relative homology with coefficients in some local system. For a cycle $\Gamma \subset Y_q$ in such a relative homology, we can define the *equivariant oscillatory integral:*

$$\int_\Gamma e^{W^\lambda/z}\omega_q = \int_\Gamma e^{W/z}\prod_{i=1}^m w_i^{\lambda_i/z}\omega_q.$$

For our purpose, it is more convenient to use the exponent $\lambda_i/(2\pi\mathbf{i})$ instead of λ_i/z. Define

$$\mathcal{I}^\lambda_\Gamma(q,z) := \frac{1}{(2\pi\mathbf{i})^n}\int_\Gamma e^{\frac{w_1+\cdots+w_m}{z}}\prod_{i=1}^m w_i^{\frac{\lambda_i}{2\pi\mathbf{i}}}\omega_q. \tag{72}$$

Again, the equivariant oscillatory integral $\mathcal{I}^\lambda_{\Gamma_{\mathbb{R}}}(q,-z)$ for the real Lefschetz thimble $\Gamma_{\mathbb{R}}$ is well-defined when $q \in \mathcal{M}_{\mathbb{R}}$ and $z > 0$.

4.4.2. Equivariant H-function

Recall that the quantum cohomology central charge can be written in terms of the H-function (33) (see (34)). Under the mirror theorem, we can write the H-function as a hypergeometric series with coefficients given by products of Gamma functions. This type of hypergeometric series has been used by Horja [44], Hosono [45] and Borisov and Horja [11].[8]

By abuse of notation, we write $H(q,z) := H(\tau(q),z)$. Using Gamma functions, we can write the I-function (59) as

$$I(q,z) = e^{\sum_{a=1}^r \overline{p}_a \log q_a/z} \sum_{d\in\mathbb{K}_{\mathrm{eff}}} \frac{q^d}{z^{\langle\hat{\rho},d\rangle}} \prod_{i=1}^m \frac{\Gamma(1-\{-\langle D_i,d\rangle\}+\overline{D}_i/z)}{\Gamma(1+\langle D_i,d\rangle+\overline{D}_i/z)} \frac{\mathbf{1}_{v(d)}}{z^{\iota_{v(d)}}}.$$

Using this expression and Conjecture 4.3, we calculate the H-function (33) as

$$\begin{aligned} H(q,z) &= (-1)^n z^{n/2} \operatorname{inv}^*(2\pi\mathbf{i})^{-\deg/2} \widehat{\Gamma}(T\mathcal{X})^{-1} z^{-\rho} z^{\mu} I(q,z) \\ &= (-1)^n \sum_{d\in\mathbb{K}_{\mathrm{eff}}} x^{\frac{\overline{p}}{2\pi\mathbf{i}}+d} \frac{\mathbf{1}_{\operatorname{inv}(v(d))}}{\prod_{i=1}^m \Gamma(1+\langle D_i,d\rangle+\frac{\overline{D}_i}{2\pi\mathbf{i}})}, \end{aligned} \tag{73}$$

where we used the fact that the $v(d)$-component of $\widehat{\Gamma}(T\mathcal{X})$ (for $d\in\mathbb{K}_{\mathrm{eff}}$) is given by $\prod_{i=1}^m \Gamma(1-\{-\langle D_i,d\rangle\}+\overline{D}_i)$ and set

$$x^{\frac{\overline{p}}{2\pi\mathbf{i}}+d} := e^{\sum_{a=1}^r (\frac{\overline{p}_a}{2\pi\mathbf{i}}+\langle p_a,d\rangle)\log x_a}, \qquad \log x_a := \log q_a - \rho_a \log z \quad \left(\text{i.e. } x_a = \frac{q_a}{z^{\rho_a}}\right).$$

We introduce T-equivariant I- and H-functions. As in Section 3.1.2, $\xi\in\mathbb{L}^\vee$ defines the orbifold line bundle L_ξ on $\mathcal{X}$:

$$L_\xi = \mathcal{U}_\eta \times \mathbb{C}/(z_1,\dots,z_m,c) \sim \left(t^{D_1}z_1,\dots,t^{D_m}z_m,t^{\xi}c\right), \quad t\in\mathbb{T}.$$

The line bundle L_ξ admits a canonical T-action: $T=(\mathbb{C}^*)^m$ acts diagonally on the first factor and the trivially on the second factor. By taking the T-equivariant first Chern class, we can associate to every element $\xi\in\mathbb{L}^\vee$ an equivariant class $c_1^T(L_\xi)\in H_T^2(\mathcal{X})$. We denote by $\overline{p}_1^\lambda,\dots,\overline{p}_r^\lambda\in H_T^2(\mathcal{X})$ the T-equivariant cohomology classes corresponding to $p_1,\dots,p_r\in\mathbb{L}^\vee$. Note that $\overline{p}_{r'+1}^\lambda,\dots,\overline{p}_r^\lambda$ may be non-zero. We denote by $\overline{D}_i^\lambda\in H_T^2(\mathcal{X})$ the *T-equivariant Poincaré dual* of the toric divisor $\{z_i=0\}$. Note that $\overline{D}_j^\lambda=0$ for $j>m'$ even in equivariant cohomology (since $\{z_j=0\}$ is empty). When $e^{-\lambda_i}$ denotes the 1-dimensional T-representation given by the ith projection $T\to\mathbb{C}^*$, the divisor $\{z_i=0\}$ becomes the zero-locus of a T-equivariant section of $L_{D_i}\otimes e^{-\lambda_i}$. Thus we have (cf. (39))

$$\overline{D}_i^\lambda = \sum_{a=1}^r \mathsf{m}_{ia}\overline{p}_a^\lambda - \lambda_i \quad \text{in } H_T^2(\mathcal{X}). \tag{74}$$

[8] We named it after Horja and Hosono.

The equivariant I-function is defined by the same formula in Definition 4.1 with all the appearance of $\overline{p}_a$, $\overline{D}_j$ replaced by $\overline{p}_a^\lambda$, $\overline{D}_j^\lambda$. The equivariant H-function $H^\lambda(q,z)$ is defined[9] to be:

$$H^\lambda(q,z) := (-1)^n z^{-\frac{\lambda_1+\cdots+\lambda_m}{2\pi\mathbf{i}}} \sum_{d\in\mathbb{K}_{\mathrm{eff}}} x^{\frac{\overline{p}^\lambda}{2\pi\mathbf{i}}+d} \frac{\mathbf{1}_{\mathrm{inv}(v(d))}}{\prod_{i=1}^m \Gamma(1+\langle D_i,d\rangle + \frac{\overline{D}_i^\lambda}{2\pi\mathbf{i}})}. \tag{75}$$

We regard the equivariant I- and H-functions as functions taking values in $H^*_{\mathrm{orb},T}(\mathcal{X})$ and $H^*_T(I\mathcal{X})$ respectively. (Here $H^*_{\mathrm{orb},T}(\mathcal{X}) := \bigoplus_{v\in\mathsf{T}} H_T^{*-2\iota_v}(\mathcal{X}_v)$.)

Remark 4.16. The equivariant I- and H-functions should be understood as follows. For a toric orbifold $\mathcal{X}$, $H^*_T(I\mathcal{X})$ is a free $H^*_T(\mathrm{pt}) = \mathbb{C}[\lambda_1,\dots,\lambda_m]$-module of rank $\dim H^*(I\mathcal{X})$. Thus we can regard $H^*_T(I\mathcal{X})$ as a finite-dimensional vector bundle over $\operatorname{Spec} H^*_T(\mathrm{pt})$. The I-function (resp. H-function) makes sense as a multi-valued meromorphic (resp. holomorphic) section of the $H^*_{\mathrm{orb}}(\mathcal{X})$-bundle over the space $\{(q,z,\lambda)\in\mathcal{M}\times\mathbb{C}^*\times\operatorname{Spec} H^*_T(\mathrm{pt});\ 0<|q_a|<\epsilon\}$.

4.4.3. Oscillatory integral and H-function

We prove a T-equivariant generalization of (70). Since $Z(\mathcal{O}_\mathcal{X})$ can be written in terms of the H-function (34), the following theorem proves (70) by the non-equivariant limit $\lambda_i\to 0$.

Theorem 4.17. *Assume that $\hat{\rho}\in\mathrm{cl}(\widetilde{C}_\mathcal{X})$. The equivariant oscillatory integral* (72) *and the equivariant H-function* (75) *are related by*

$$\mathcal{I}^\lambda_{\Gamma_\mathbb{R}}(q,-z) = \int_{I\mathcal{X}} H^\lambda\big(q,e^{\pi\mathbf{i}}z\big)\cup\widetilde{\mathrm{Td}}^\lambda(T\mathcal{X}),\quad q\in\mathcal{M}_\mathbb{R},\ z>0, \tag{76}$$

where $\widetilde{\mathrm{Td}}^\lambda(T\mathcal{X})$ is the T-equivariant Todd class defined similarly to Section 2.4. *The branches of the logarithm in the right-hand side are chose so that* $\log z\in\mathbb{R}$, $\log q_a\in\mathbb{R}$.

Remark 4.18. Even if $\hat{\rho}\notin\mathrm{cl}(\widetilde{C}_\mathcal{X})$, the left-hand side of (76) makes sense as an analytic function in q and z. In this case, the right-hand side could be understood as the asymptotic expansion in $q_1,\dots,q_r$ of the left-hand side in the limit $q_a\searrow +0$.

By the localization theorem [5] in equivariant cohomology, the inclusion $i: I\mathcal{X}^T\to I\mathcal{X}$ of the T-fixed point set $I\mathcal{X}^T$ induces an isomorphism $i^*: H^*_T(I\mathcal{X})\otimes_{H^*_T(\mathrm{pt})}\mathbb{C}(\lambda)\to H^*(I\mathcal{X}^T)\otimes_{H^*_T(\mathrm{pt})}\mathbb{C}(\lambda)$, where $\mathbb{C}(\lambda)$ is the fraction field of $H^*_T(\mathrm{pt})=\mathbb{C}[\lambda_1,\dots,\lambda_m]$. The number of fixed points in $I\mathcal{X}$ is equal to $N:=\dim H^*_{\mathrm{orb}}(\mathcal{X})$ (see the proof of Lemma 3.9). A T-fixed point in $I\mathcal{X}$ is labeled by a pair (σ,v) of a fixed point $\sigma\in\mathcal{X}^T$ and $v\in\mathrm{Box}$ such that $\sigma\in\mathcal{X}_v$. Moreover, a fixed point $\sigma\in\mathcal{X}^T$ is in one-to-one correspondence with a maximal cone of the fan Σ spanned by $\{b_i;\ \sigma\in\{z_i=0\}\}$. By restricting $H^\lambda(q,z)$ to a fixed point (σ,v), we get a function $H^\lambda_{\sigma,v}(q,z)$ in q, z and λ. We call it a *component* of the H-function.

Lemma 4.19. *The equivariant H-function $H^\lambda(q,z)$ and the oscillatory integral $\mathcal{I}^\lambda_{\Gamma_\mathbb{R}}(q,z)$ are solutions to the following GKZ-type differential equations:*

[9] The factor $z^{-\frac{\lambda_1+\cdots+\lambda_m}{2\pi\mathbf{i}}}$ comes from the T-equivariant first Chern class $c_1^T(T\mathcal{X}) = \sum_{a=1}^{r'}\rho_a\overline{p}_a^\lambda - (\lambda_1+\cdots+\lambda_m)$.

$$\mathcal{P}_d^\lambda f(q,z) = 0, \quad d \in \mathbb{L}, \tag{77}$$

$$\left(z\frac{\partial}{\partial z} + \sum_{a=1}^r \rho_a \partial_a\right) f(q,z) = \frac{\lambda_1 + \cdots + \lambda_m}{2\pi\mathbf{i}} f(q,z), \tag{78}$$

where $\partial_a := q_a(\partial/\partial q_a)$*,*

$$\mathcal{P}_d^\lambda := q^d \prod_{\langle D_i,d\rangle<0} \prod_{\nu=0}^{-\langle D_i,d\rangle-1} \left(\mathcal{D}_i^\lambda - \nu z\right) - \prod_{\langle D_i,d\rangle>0} \prod_{\nu=0}^{\langle D_i,d\rangle-1} \left(\mathcal{D}_i^\lambda - \nu z\right),$$

and $\mathcal{D}_i^\lambda := \sum_{a=1}^r \mathsf{m}_{ia} z\partial_a - z\lambda_i/(2\pi\mathbf{i})$*. The N components $H_{\sigma,v}^\lambda(q,z)$ of the H-function form a basis of solutions to these differential equations for generic λ_i's and small q_a's.*

Proof. The proof here is an equivariant generalization of the argument in Section 4.2. The proof of (77) for $f(q,z) = H^\lambda(q,z)$ or $\mathcal{I}_{\Gamma_\mathbb{R}}^\lambda(q,z)$ is similar to Lemma 4.6. (For $H^\lambda(q,z)$, we rewrite (75) as a summation over $d \in \mathbb{K}$; the terms with $d \in \mathbb{K} \setminus \mathbb{K}_{\mathrm{eff}}$ automatically vanish by relations in $H_T^*(I\mathcal{X})$.) Eq. (78) means the homogeneity of $f(q,z)$. The details are left to the reader.

In order to show that the components of the H-function form a basis of solutions, we consider the *equivariant GKZ D-module:*

$$M_{\mathrm{GKZ}}^\lambda := \mathbb{C}\left[z,q^\pm\right]\langle z\partial\rangle \Big/ \sum_{d\in\mathbb{L}} \mathbb{C}\left[z,q^\pm\right]\langle z\partial\rangle \mathcal{P}_d^\lambda$$

for fixed complex numbers $\lambda_1,\dots,\lambda_m$. This also admits a flat connection as in Section 4.2. Since the differential operator $\mathcal{P}_d^\lambda$ has the same principal symbol as $\mathcal{P}_d$ and $M_{\mathrm{GKZ}}^\lambda/zM_{\mathrm{GKZ}}^\lambda$ is independent of λ, the same argument as the proof of Proposition 4.4 shows that $M_{\mathrm{GKZ}}^\lambda \otimes_{\mathbb{C}[z,q^\pm]} \mathcal{O}_{\mathcal{M}^\circ\times\mathbb{C}^*}$ is locally free of rank $\leqslant N$. Therefore, we have at most N linearly independent solutions to the GKZ-system (77), (78). On the other hand, similarly to Lemma 4.7, we can show that

$$\begin{aligned}&\left(q^{-\delta} \prod_{i=1}^m \prod_{\nu=0}^{\lceil\langle D_i,\delta\rangle\rceil-1} \left(\mathcal{D}_i^\lambda - \nu z\right)\right) H^\lambda(q,z) \\ &\quad = (-1)^n z^{\frac{\lambda_1+\cdots+\lambda_m}{2\pi\mathbf{i}} + \iota_{v(\delta)}} x^{\frac{\overline{p}^\lambda}{2\pi\mathbf{i}}} \left(\frac{\mathbf{1}_{\mathrm{inv}(v(\delta))}}{\prod_{i=1}^m \Gamma(1-\{-\langle D_i,\delta\rangle\} + \frac{\overline{D}_i^\lambda}{2\pi\mathbf{i}})} + O\left(q^{1/e_0}\right)\right)\end{aligned}$$

for $\delta \in \mathbb{K}$ such that $\langle D_i,\delta\rangle > 0$ for all i. Because $H_T^*(I\mathcal{X})$ is generated by $\mathbf{1}_v$, $v \in \mathrm{Box}$ over $H_T^*(\mathrm{pt})[\overline{p}_1^\lambda,\dots,\overline{p}_{r'}^\lambda]$ (cf. (42), (43)), suitable derivatives of $H^\lambda(q,z)$ form a meromorphic basis[10] of $H_T^*(I\mathcal{X})$ (cf. (64)). This shows that N components $H_{\sigma,v}^\lambda(q,z)$ of $H^\lambda(q,z)$ are linearly independent for generic values of $\lambda_1,\dots,\lambda_m$. □

From this lemma, we know that there exist coefficient functions $c_{\sigma,v}(\lambda)$ such that

$$\mathcal{I}_{\Gamma_\mathbb{R}}^\lambda(q,-z) = \sum_{(\sigma,v)\in I\mathcal{X}^T} c_{\sigma,v}(\lambda) H_{\sigma,v}^\lambda\left(q,e^{\pi\mathbf{i}}z\right). \tag{79}$$

[10] Here we regard $H_T^*(I\mathcal{X})$ as a vector bundle over $\mathrm{Spec}\, H_T^*(\mathrm{pt})$ as in Remark 4.16.

We will determine an analytic function $c_{\sigma,v}(\lambda)$ in λ by putting $z=1$ and studying the asymptotic behavior of the both-hand sides in the limit $q_a \searrow +0$.

We start with the oscillatory integral. Take a fixed point $\sigma \in \mathcal{X}^T$. Define $I^\sigma \in \mathcal{A}$ by $I^\sigma = \{i;\ \sigma \notin \{z_i = 0\}\}$. We can take $\{w_j;\ j \notin I^\sigma\}$ as a co-ordinate system on $Y_q \cap Y_\mathbb{R} = \Gamma_\mathbb{R}$. We can express w_i for $i \in I^\sigma$ in terms of $\{w_j;\ j \notin I^\sigma\}$ and q_a, $a = 1, \dots, r$ by solving (47). Put

$$w_i = \prod_{a=1}^{r} q_a^{\ell_{ia}^\sigma} \prod_{j \notin I^\sigma} w_j^{b_{ij}^\sigma}, \quad i \in I^\sigma.$$

Here $(\ell_{ia}^\sigma)_{i \in I^\sigma, 1 \leqslant a \leqslant r}$ is the matrix inverse to $(\mathsf{m}_{ia})_{i \in I^\sigma, 1 \leqslant a \leqslant r}$. Because $p_a \in \mathrm{cl}(\widetilde{C}_\mathcal{X}) \subset \sum_{i \in I^\sigma} \mathbb{R}_{\geqslant 0} D_i$, it follows that $\ell_{ia}^\sigma \geqslant 0$. We can see that $b_{ij}^\sigma \in \mathbb{Q}$ is determined by $b_i = \sum_{j \notin I^\sigma} b_{ij}^\sigma b_j$ in $\boldsymbol{N} \otimes \mathbb{R}$. Let $V(\sigma)$ be $n!|\boldsymbol{N}_{\mathrm{tor}}|$ times the volume of the convex hull of $\{b_j;\ j \notin I^\sigma\} \cup \{0\}$ in $\boldsymbol{N} \otimes \mathbb{R}$. The holomorphic volume form ω_q can be written in terms of $\{w_j;\ j \notin I^\sigma\}$ as

$$\omega_q = \frac{1}{V(\sigma)} \prod_{j \notin I^\sigma} \frac{dw_j}{w_j}.$$

We set

$$\mathbb{K}_{\mathrm{eff},\sigma} := \left\{ d \in \mathbb{L} \otimes \mathbb{Q};\ \langle D_i, d\rangle \in \mathbb{Z}_{\geqslant 0},\ \forall i \in I^\sigma \right\} = \bigoplus_{i \in I^\sigma} \mathbb{Z}_{\geqslant 0} \ell_i^\sigma.$$

Here, $\ell_i^\sigma \in \mathbb{L} \otimes \mathbb{Q}$ is defined by $\langle p_a, \ell_i^\sigma \rangle = \ell_{ia}^\sigma$. Then we have $\mathbb{K}_{\mathrm{eff}} = \bigcup_{\sigma \in \mathcal{X}^T} \mathbb{K}_{\mathrm{eff},\sigma}$. We denote by $\overline{p}_a^\lambda(\sigma)$ and $\overline{D}_j^\lambda(\sigma)$ the restrictions of $\overline{p}_a^\lambda, \overline{D}_j^\lambda \in H_T^*(\mathcal{X})$ to the fixed point σ. By using $\overline{D}_i^\lambda(\sigma) = 0$ for $i \in I^\sigma$ and (74), we calculate

$$\overline{p}_a^\lambda(\sigma) = \sum_{i \in I^\sigma} \lambda_i \ell_{ia}^\sigma, \qquad \overline{D}_j^\lambda(\sigma) = -\lambda_j - \sum_{i \in I^\sigma} \lambda_i b_{ij}^\sigma, \quad j \notin I^\sigma. \tag{80}$$

For a function $f(q_1, \dots, q_r)$ in $(q_1, \dots, q_r) \in (\mathbb{R}_{>0})^r$, we write $f(q_1, \dots, q_r) = O(M)$ for $M \in \mathbb{R}$ when $f(tq_1, \dots, tq_r) = O(t^M)$ as $t \searrow +0$.

Lemma 4.20. *Let σ be a fixed point in $\mathcal{X}$. For any $M > 0$, there exists $M' > 0$ such that the following holds. For $\lambda_1, \dots, \lambda_m$ such that $\Re(-\overline{D}_j^\lambda(\sigma)/(2\pi\mathbf{i})) > M'$ for all $j \notin I^\sigma$, $\mathcal{I}_{\Gamma_\mathbb{R}}^\lambda(q, -1)$ with $(q_1, \dots, q_r) \in (\mathbb{R}_{>0})^r$ has the expansion*

$$\begin{aligned}
&\mathcal{I}_{\Gamma_\mathbb{R}}^\lambda(q, -1) \\
&\quad = (-1)^n \frac{e^{(\lambda_1 + \cdots + \lambda_m)/2}}{V(\sigma)} \left(e^{-\pi\mathbf{i}\hat{\rho}} q\right)^{\frac{\overline{p}^\lambda(\sigma)}{2\pi\mathbf{i}}} \\
&\qquad \times \left(\sum_{\substack{d \in \mathbb{K}_{\mathrm{eff},\sigma} \\ |d| < M}} \frac{(e^{-\pi\mathbf{i}\hat{\rho}} q)^d}{\prod_{j \notin I^\sigma} (1 - e^{-2\pi\mathbf{i}\langle D_j, d\rangle - \overline{D}_j^\lambda(\sigma)}) \prod_{i=1}^m \Gamma(1 + \langle D_i, d\rangle + \frac{\overline{D}_i^\lambda(\sigma)}{2\pi\mathbf{i}})} + O(M) \right),
\end{aligned}$$

where $|d| := \sum_{a=1}^r \langle p_a, d\rangle$ and we set

$$\big(e^{-\pi\mathbf{i}\hat{\rho}}q\big)^{\frac{\overline{p}^{\lambda}(\sigma)}{2\pi\mathbf{i}}} := \prod_{a=1}^{r}\big(e^{-\pi\mathbf{i}\rho_a}q_a\big)^{\frac{\overline{p}_a^{\lambda}(\sigma)}{2\pi\mathbf{i}}}, \qquad \big(e^{-\pi\mathbf{i}\hat{\rho}}q\big)^{d} := \prod_{a=1}^{r}\big(e^{-\pi\mathbf{i}\rho_a}q_a\big)^{\langle p_a, d\rangle}.$$

Proof. Using the notation above and (80), we can write

$$\mathcal{I}^{\lambda}_{\Gamma_{\mathbb{R}}}(q,-1) = \frac{q^{\frac{\overline{p}^{\lambda}(\sigma)}{2\pi\mathbf{i}}}}{(2\pi\mathbf{i})^n V(\sigma)} \int_{(0,\infty)^n} \exp\Big(-\sum_{i\in I^{\sigma}} q^{\ell_i^{\sigma}} w_{\sigma}^{b_i}\Big) e^{-\sum_{j\notin I^{\sigma}} w_j} w_{\sigma}^{-\frac{\overline{D}^{\lambda}(\sigma)}{2\pi\mathbf{i}}} \frac{dw_{\sigma}}{w_{\sigma}},$$

where we put

$$w_{\sigma}^{b_i} := \prod_{j\notin I^{\sigma}} w_j^{b_{ij}^{\sigma}}, \qquad w_{\sigma}^{-\frac{\overline{D}^{\lambda}(\sigma)}{2\pi\mathbf{i}}} := \prod_{j\notin I^{\sigma}} w_j^{-\frac{\overline{D}_j^{\lambda}(\sigma)}{2\pi\mathbf{i}}} \quad \text{and} \quad \frac{dw_{\sigma}}{w_{\sigma}} := \prod_{j\notin I^{\sigma}} \frac{dw_j}{w_j}.$$

Consider the Taylor expansion:

$$\exp\Big(-\sum_{i\in I^{\sigma}} q^{\ell_i^{\sigma}} w_{\sigma}^{b_i}\Big) = \sum_{\substack{n_i\geqslant 0;\ i\in I^{\sigma}\\ |\sum_{i\in I^{\sigma}} n_i\ell_i^{\sigma}|<M}} \frac{\prod_{i\in I^{\sigma}}(-1)^{n_i} q^{n_i\ell_i^{\sigma}} w_{\sigma}^{n_i b_i}}{\prod_{i\in I^{\sigma}} n_i!} + O(M).$$

When $\Re(-\overline{D}_j^{\lambda}(\sigma)/(2\pi\mathbf{i}))$ is sufficiently big for all $j\notin I^{\sigma}$, each term in the right-hand side is integrable for the measure $e^{-\sum_{j\notin I^{\sigma}} w_j} w_{\sigma}^{-\frac{\overline{D}^{\lambda}(\sigma)}{2\pi\mathbf{i}}}(dw_{\sigma}/w_{\sigma})$ on $(0,\infty)^n$. Therefore, we calculate

$$\begin{aligned}\mathcal{I}^{\lambda}_{\Gamma_{\mathbb{R}}}(q,-1) &= \frac{q^{\frac{\overline{p}^{\lambda}(\sigma)}{2\pi\mathbf{i}}}}{(2\pi\mathbf{i})^n V(\sigma)} \\ &\quad\times\Bigg(\sum_{\substack{d\in\mathbb{K}_{\mathrm{eff},\sigma},\\ |d|<M}} \frac{(-1)^{\sum_{i\in I^{\sigma}} n_i} q^d}{\prod_{i\in I^{\sigma}} n_i!} \prod_{j\notin I^{\sigma}} \Gamma\Big(\sum_{i\in I^{\sigma}} n_i b_{ij}^{\sigma} - \frac{\overline{D}_j^{\lambda}(\sigma)}{2\pi\mathbf{i}}\Big) + O(M)\Bigg),\end{aligned}$$

where $d=\sum_{i\in I^{\sigma}} n_i\ell_i^{\sigma}$. Using $n_i=\langle D_i,d\rangle$, $\sum_{i\in I^{\sigma}} n_i b_{ij}^{\sigma} = -\langle D_j,d\rangle$ and $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$, we arrive at the formula in the lemma. $\square$

Next we study the asymptotics of $H^{\lambda}_{\sigma,v}(q,e^{\pi\mathbf{i}})$ in the limit $q\searrow +0$.

Lemma 4.21. *Let σ be a fixed point in $\mathcal{X}$. For a given $M>0$, there exists an open set $V\subset(\mathbb{C})^m$ such that both the expansion in Lemma* 4.20 *and the expansion*

$$H^{\lambda}_{\tau,v}\big(q,e^{\pi\mathbf{i}}\big) = (-1)^n e^{(\lambda_1+\cdots+\lambda_m)/2}\big(e^{-\pi\mathbf{i}\hat{\rho}}q\big)^{\frac{\overline{p}^{\lambda}(\sigma)}{2\pi\mathbf{i}}} \times \begin{cases} \displaystyle\sum_{\substack{d\in\mathbb{K}_{\mathrm{eff},\sigma}\\ \mathrm{inv}(v(d))=v,|d|<M}} \frac{(e^{-\pi\mathbf{i}\hat{\rho}}q)^d}{\prod_{i=1}^m \Gamma(1+\langle D_i,d\rangle+\frac{\overline{D}_i^{\lambda}(\sigma)}{2\pi\mathbf{i}})} + O(M) & \text{if } \tau=\sigma; \\ O(M) & \text{if } \tau\neq\sigma \end{cases}$$

hold when $(\lambda_1,\ldots,\lambda_m)\in V$.

Proof. By the definition (75) of $H^\lambda(q,z)$, it suffices to show that both the expansion in Lemma 4.20 and the inequality

$$\sum_{a=1}^{r} \Re\left(\frac{\overline{p}_a^\lambda(\sigma)}{2\pi\mathbf{i}}\right) + M < \sum_{a=1}^{r} \Re\left(\frac{\overline{p}_a^\lambda(\tau)}{2\pi\mathbf{i}}\right), \quad \forall\tau \neq \sigma, \tag{81}$$

hold for some $(\lambda_1,\dots,\lambda_m) \in (\mathbf{i}\mathbb{R})^m$. (Note that these are open conditions for λ.) Hereafter we take λ_j to be purely imaginary. Recall that $\mathcal{X}$ can be written as a symplectic quotient $\mathfrak{h}^{-1}(\eta)/\mathbb{T}^r_{\mathbb{R}}$ (35) and is endowed with the reduced symplectic form depending on η. Without changing the orbifold $\mathcal{X}$, we can choose the vector $\eta \in \mathbb{L}\otimes\mathbb{R}$ to be $p_1+\cdots+p_r \in \widetilde{C}_{\mathcal{X}}$. Define a Hamiltonian function $\mathfrak{h}_{\eta,\lambda} : \mathcal{X} \to \mathbb{R}$ by

$$\mathfrak{h}_{\eta,\lambda}(z_1,\dots,z_m) = -\sum_{i=1}^{m} \frac{\lambda_j}{2\pi\mathbf{i}}|z_j|^2, \quad (z_1,\dots,z_m) \in \mathfrak{h}^{-1}(\eta).$$

This generates an (almost periodic) Hamiltonian $\mathbb{R}$-action $z_i \mapsto e^{-\lambda_i s}z_i$, $s \in \mathbb{R}$ on $\mathcal{X}$. In general, an almost periodic Hamiltonian attains its global maximum at a critical point of index $2n = \dim_{\mathbb{R}}\mathcal{X}$. (This follows from the so-called Mountain-Path Lemma and the fact that there are no critical points of odd index. See e.g. [6].) Because the weights of $T_\sigma\mathcal{X}$ for this $\mathbb{R}$-action are $\{\overline{D}_j^\lambda(\sigma)/(2\pi\mathbf{i});\ j \notin I^\sigma\}$, it follows that

$$-\overline{D}_j^\lambda(\sigma)/(2\pi\mathbf{i}) > 0, \quad \forall j \notin I^\sigma \quad \Longrightarrow \quad \mathfrak{h}_{\eta,\lambda} \text{ attains its unique maximum at } \sigma. \tag{82}$$

By (80), one can choose $(\lambda_1,\dots,\lambda_m) \in (\mathbf{i}\mathbb{R})^m$ so that $-\overline{D}_j^\lambda(\sigma)/(2\pi\mathbf{i})$, $j \notin I^\sigma$, are arbitrarily large positive numbers and that the expansion in Lemma 4.20 holds. Then by (82), we know that $\mathfrak{h}_{\eta,\lambda}(\sigma) > \mathfrak{h}_{\eta,\lambda}(\tau)$ for every other fixed point $\tau \neq \sigma$. On the other hand, using $\eta = p_1+\cdots+p_r$, we can easily show that $\mathfrak{h}_{\eta,\lambda}(\sigma) = -\sum_{a=1}^r \overline{p}_a^\lambda(\sigma)/(2\pi\mathbf{i})$. Therefore, by rescaling λ_i if necessary, we can achieve the inequality (81). $\square$

Comparing the expansions in Lemmas 4.20 and 4.21, we conclude

$$c_{\sigma,v}(\lambda) = \frac{1}{V(\sigma)\prod_{i\notin I^\sigma}(1-e^{-2\pi\mathbf{i}f_v([D_i])-\overline{D}_i^\lambda(\sigma)})},$$

where $c_{\sigma,v}$ is the coefficient appearing in (79) and $f_v([D_i]) \in [0,1)$ is the rational number associated to $[D_i] \in H^2(\mathcal{X},\mathbb{Z})$ (see Section 2 and (44)). Hence, we find

$$c_{\sigma,v}(\lambda) = \frac{1}{V(\sigma)} \frac{\widetilde{\mathrm{Td}}^\lambda(T\mathcal{X})|_{(\sigma,v)}}{e_T(T_\sigma\mathcal{X}_v)},$$

where $\widetilde{\mathrm{Td}}^\lambda(T\mathcal{X})|_{(\sigma,v)}$ is the restriction of the equivariant Todd class $\widetilde{\mathrm{Td}}^\lambda(T\mathcal{X})$ to the fixed point (σ,v) in $I\mathcal{X}$ and $e_T(T_\sigma\mathcal{X}_v)$ is the T-equivariant Euler class of $T_\sigma\mathcal{X}_v$ (regarded as a T-equivariant vector bundle over a point σ). Here λ_i is regarded as an element of $H_T^2(\mathrm{pt})$ and we used the fact that

$$e_T(T_\sigma \mathcal{X}_v) = \prod_{i \notin I^\sigma, f_v([D_i])=0} \overline{D}_i^\lambda(\sigma).$$

Since $V(\sigma)$ is the order of the automorphism group $\mathrm{Aut}(\sigma)$ at $\sigma \in \mathcal{X}$, Eq. (76) follows from the localization theorem in T-equivariant cohomology [5].

4.4.4. Proof of (71)

We use (71) when we prove the matching of pairings in Appendix A.3. Since the both-hand sides of (71) are monodromy-invariant, by (34), it suffices to prove that

$$(-1)^n i_{\mathrm{pt}}^*\big(H(q,z)\big) = \frac{1}{(2\pi\mathbf{i})^n} \int_{\Gamma_{\mathrm{c}}} e^{W_q/z} \omega_q$$

where $i_{\mathrm{pt}} : \mathrm{pt} \to \mathcal{X} \subset I\mathcal{X}$ is an inclusion of a point and we used the fact that $[\mathcal{O}_{\mathrm{pt}}^\vee] = (-1)^n[\mathcal{O}_{\mathrm{pt}}]$. By the residue calculations, the right-hand side is (see (48)):

$$\sum_{\substack{(k_1,\dots,k_m)\in(\mathbb{Z}_{\geqslant 0})^m \\ \sum_{i=1}^m k_i b_i = 0}} \frac{1}{k_1! \cdots k_m!} \frac{q^{k_1\ell_1+\cdots+k_m\ell_m}}{z^{k_1+\cdots+k_m}}.$$

Because $(k_1, \dots, k_m)$ appearing in the summation gives an element $d \in \mathbb{L}$ such that $k_i = \langle D_i, d\rangle$, we can see that this equals $(-1)^n i_{\mathrm{pt}}^* H(q,z)$ by (73).

5. Integral periods and crepant resolution conjecture

In mirror symmetry for Calabi–Yau manifolds (see e.g. [16,29,59]), flat co-ordinates (or mirror map) τ_i on the B-model in a neighborhood of a maximally unipotent monodromy point was given by periods over integral cycles $A_1, \dots, A_r$ of a holomorphic n-form Ω

$$\tau_i = \int_{A_i} \Omega,$$

where Ω is normalized by the condition:

$$\int_{A_0} \Omega = 1.$$

Here, A_0 is a monodromy-invariant cycle (unique up to sign) and $A_1, \dots, A_r$ are such that they transforms under monodromy as $A_i \mapsto A_i + k_i A_0$. Thus, in Calabi–Yau case, *flat co-ordinates are constructed as integral periods*.

In this section, we consider integral periods in the A-model by choosing some integral structure on it. The integral structure in this section does not need to be the $\widehat{\Gamma}$-integral structure. We study relationships between integral periods and flat co-ordinates in the conformal limit (85). Then we discuss why quantum parameters should be specialized to roots of unity in Y. Ruan's crepant resolution conjecture [66]. Throughout this section, we assume that $\mathcal{X}$ is a weak Fano (i.e. $\rho = c_1(\mathcal{X})$ is nef) Gorenstein projective orbifold without generic stabilizer.

5.1. Integral periods in the A-model

In what follows, we fix an integral lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ in the space $\mathcal{S}(\mathcal{X})$ of flat sections of the quantum D-module $QDM(\mathcal{X})$ satisfying

- $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ is invariant under the Galois action: $G^{\mathcal{S}}(\xi)\mathcal{S}(\mathcal{X})_{\mathbb{Z}} = \mathcal{S}(\mathcal{X})_{\mathbb{Z}}$,
- the pairing $(\cdot,\cdot)_{\mathcal{S}}$ restricts to a $\mathbb{Z}$-valued pairing: $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} \times \mathcal{S}(\mathcal{X})_{\mathbb{Z}} \to \mathbb{Z}$.

An example is given by the $\widehat{\Gamma}$-integral structure $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} = \mathcal{Z}_K(K(\mathcal{X}))$. (See Definition 2.9, Proposition 2.10.) An *integral period* in the A-model is defined to be a pairing between a section of $QDM(\mathcal{X})$ and an element of $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$. The quantum cohomology central charge (25) is an example of integral periods.

We set up the notation. We set $\mathcal{V} := H^*_{\mathrm{orb}}(\mathcal{X})$. This is identified with $\mathcal{S}(\mathcal{X})$ via the cohomology framing $\mathcal{Z}_{\mathrm{coh}} : \mathcal{V} = H^*_{\mathrm{orb}}(\mathcal{X}) \to \mathcal{S}(\mathcal{X})$ in (19). The integral structure $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ induces an integral lattice $\mathcal{V}_{\mathbb{Z}}$ in $\mathcal{V}$:

$$\mathcal{V}_{\mathbb{Z}} := \mathcal{Z}_{\mathrm{coh}}^{-1}\big(\mathcal{S}(\mathcal{X})_{\mathbb{Z}}\big) \subset \mathcal{V} = H^*_{\mathrm{orb}}(\mathcal{X}).$$

For $A \in \mathcal{V}_{\mathbb{Z}}$ and $\alpha \in H^*_{\mathrm{orb}}(\mathcal{X})$, we put

$$\Pi^{\alpha}_{A}(\tau, z) := \big(\alpha, \mathcal{Z}_{\mathrm{coh}}(A)(\tau, z)\big)_{\mathrm{orb}} = \big(L(\tau, -z)^{-1}\alpha, z^{-\mu} z^{\rho} A\big)_{\mathrm{orb}}, \tag{83}$$

where we used $L(\tau, -z)^{-1} = L(\tau, z)^{\dagger}$. The quantum cohomology central charge (25) is given by $Z(V) = c(z)\Pi^{\mathbf{1}}_{\Psi(V)}$. (We do not need the $\widehat{\Gamma}$-class to define Π^{α}_{A}.)

In order to consider the integral periods (83) without $\log z$ terms, we introduce the sublattice $\mathcal{V}_{\mathbb{Z},\rho} \subset \mathcal{V}_{\mathbb{Z}}$ by

$$\mathcal{V}_{\mathbb{Z},\rho} := \mathrm{Ker}(\rho) \cap \mathcal{V}_{\mathbb{Z}}.$$

By the assumption that $\mathcal{X}$ is Gorenstein, all the ages ι_v are integers and $H^*_{\mathrm{orb}}(\mathcal{X})$ is graded by even integers. Therefore, an element of $\mathcal{V}_{\mathbb{Z},\rho}$ corresponds to a flat section which is single-valued (when $n = \dim_{\mathbb{C}} \mathcal{X}$ is even) or two-valued (when n is odd) under $\mathcal{Z}_{\mathrm{coh}}$. We write the integral period Π^{α}_{A} for $A \in \mathcal{V}_{\mathbb{Z},\rho}$ as a pairing on the "two-valued Givental space" $\widehat{\mathcal{H}}$:

$$\widehat{\mathcal{H}} := \mathcal{H} \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}\big(\mathbb{C}^*_{z^{1/2}}\big),$$

where $\mathbb{C}^*_{z^{1/2}} \to \mathbb{C}^*$ denotes the double cover of the z-plane. The pairing (27) on $\mathcal{H}$ is naturally extended to $\widehat{\mathcal{H}}$ as

$$\big(\alpha\big(z^{1/2}\big), \beta\big(z^{1/2}\big)\big)_{\mathcal{H}} = \big(\alpha\big(\mathbf{i}z^{1/2}\big), \beta\big(z^{1/2}\big)\big)_{\mathrm{orb}}.$$

Then we have for $A \in \mathcal{V}_{\mathbb{Z},\rho}$

$$\Pi^{\alpha}_{A}(\tau, z) = \big(\mathbb{J}_{\tau}\alpha, z^{-\mu} A\big)_{\mathcal{H}}, \tag{84}$$

where $\mathbb{J}_{\tau}\alpha = L(\tau, z)^{-1}\alpha$ is given in (31). Recall that $\mathbb{J}_{\tau}\alpha$ is lying on the semi-infinite Hodge structure $\mathbb{F}_{\tau}$ in Section 2.5.

5.2. Conformal limit and integral periods

By *conformal limit* we mean the following limit sequence in $H^2_{\mathrm{orb}}(\mathcal{X})$:

$$\tau - s\rho, \qquad \Re(s) \to \infty \tag{85}$$

with a fixed $\tau \in H^2_{\mathrm{orb}}(\mathcal{X})$. Using the assumption that $\rho = c_1(\mathcal{X})$ is nef, we can define the conformal limit of $\mathbb{J}_\tau = L(\tau, z)^{-1}$ as follows:

$$\begin{aligned} \mathbb{J}^{\mathrm{c}}_\tau \alpha &:= \lim_{\Re(s)\to\infty} e^{s\rho/z} \mathbb{J}_{\tau - s\rho}\alpha \\ &= e^{\tau_{0,2}/z}\Biggl(\alpha + \sum_{\substack{(d,l)\neq(0,0)\\ d\in\mathrm{Ker}(\rho)}} \sum_{i=1}^{N} \frac{1}{l!}\biggl\langle \alpha, \tau_{\mathrm{tw}}, \ldots, \tau_{\mathrm{tw}}, \frac{\phi_i}{z-\psi}\biggr\rangle^{\mathcal{X}}_{0,l+2,d} e^{\langle \tau_{0,2}, d\rangle}\phi^i\Biggr). \end{aligned} \tag{86}$$

Here we put $\tau = \tau_{0,2} + \tau_{\mathrm{tw}}$ with $\tau_{0,2} \in H^2(\mathcal{X})$ and $\tau_{\mathrm{tw}} \in \bigoplus_{\iota_v=1} H^0(\mathcal{X}_v)$ and used (31) and the fact that $\langle \rho, d\rangle \geqslant 0$ for all $d \in \mathrm{Eff}_{\mathcal{X}}$. When $\alpha \in H^{2k}_{\mathrm{orb}}(\mathcal{X})$, $\mathbb{J}^{\mathrm{c}}_\tau(\alpha)$ is homogeneous of degree $2k$ if we set $\deg(z) = 2$.

Definition 5.1. Assume that $\rho = c_1(\mathcal{X})$ is nef. Let $\tau \mapsto \mathbb{F}_\tau$ be the quantum cohomology $\frac{\infty}{2}$VHS in Section 2.5. The *conformal limit quantum cohomology $\frac{\infty}{2}$VHS* is defined to be

$$\mathbb{F}^{\mathrm{c}}_\tau := \lim_{\Re(s)\to\infty} e^{s\rho/z}\mathbb{F}_{\tau-s\rho} = \mathbb{J}^{\mathrm{c}}_\tau\bigl(H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathcal{O}(\mathbb{C}^*)\bigr), \quad \tau \in H^2_{\mathrm{orb}}(\mathcal{X}).$$

This satisfies $\mathbb{F}^{\mathrm{c}}_{\tau + a\rho} = e^{a\rho/z}\mathbb{F}^{\mathrm{c}}_\tau$ and is homogeneous $(z\partial_z + \mu)\mathbb{F}^{\mathrm{c}}_\tau \subset \mathbb{F}^{\mathrm{c}}_\tau$.

Remark 5.2. The new $\frac{\infty}{2}$VHS $\mathbb{F}^{\mathrm{c}}_\tau$ can be also defined in terms of the "conformal quantum product" $\lim_{\Re(s)\to\infty} \circ_{\tau-s\rho}$ and the Dubrovin connection associated to it. This conformal limit of quantum cohomology is closely related to Y. Ruan's quantum corrected ring [66], which is defined by counting rational curves contained in the exceptional locus (in the case of crepant resolution). The conformal limit of a $\frac{\infty}{2}$VHS appears in the work of Sabbah [67, Part I] as the associated graded of a free $\mathbb{C}[z]$-module G_k (an algebraization of $z^{-k}\mathbb{F}_\tau$) with respect to the Kashiwara–Malgrange V-filtration at $z = \infty$. See also Hertling and Sevenheck [42, Section 7] for a review.

In the conformal limit, the $\frac{\infty}{2}$VHS reduces to a *finite-dimensional VHS*. We define subspaces H_0, $\widehat{\mathbb{F}}^{\mathrm{c}}_\tau$ of $\widehat{\mathcal{H}}$ by

$$H_0 := \mathrm{Ker}(z\partial_z + \mu), \qquad \widehat{\mathbb{F}}^{\mathrm{c}}_\tau := \mathbb{F}^{\mathrm{c}}_\tau \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}\bigl(\mathbb{C}^*_{z^{1/2}}\bigr).$$

The pairing $(\cdot,\cdot)_{\mathcal{H}}$ on $\widehat{\mathcal{H}}$ induces a $(-1)^n$-symmetric $\mathbb{C}$-valued pairing $(\cdot,\cdot)_{H_0}$ on H_0. The semi-infinite flag $\cdots \supset z^{-1}\widehat{\mathbb{F}}^{\mathrm{c}}_\tau \supset \widehat{\mathbb{F}}^{\mathrm{c}}_\tau \supset z\widehat{\mathbb{F}}^{\mathrm{c}}_\tau \supset \cdots$ restricts to a finite-dimensional flag $H_0 = \mathscr{F}^0_\tau \supset \mathscr{F}^1_\tau \supset \cdots \supset \mathscr{F}^n_\tau \supset 0$:

$$\mathscr{F}^p_\tau := z^{p-n/2}\widehat{\mathbb{F}}^{\mathrm{c}}_\tau \cap H_0 = \mathrm{Span}_{\mathbb{C}}\bigl\{z^{p-n/2}\mathbb{J}^{\mathrm{c}}_\tau\bigl(z^j\alpha\bigr);\ \alpha \in H^{2n-2p-2j}_{\mathrm{orb}}(\mathcal{X}),\ j \geqslant 0\bigr\}$$

satisfying the Griffiths transversality and Hodge–Riemann bilinear relation:

$$\frac{\partial}{\partial t^i}\mathscr{F}_\tau^p \subset \mathscr{F}_\tau^{p-1}, \qquad \left(\mathscr{F}_\tau^p, \mathscr{F}_\tau^{n-p+1}\right)_{H_0} = 0.$$

Conversely, the finite-dimensional VHS $\mathscr{F}_\tau^\bullet$ recovers $\mathbb{F}_\tau^{\mathrm{c}}$ by

$$\mathbb{F}_\tau^{\mathrm{c}} = z^{-n/2}\mathscr{F}_\tau^n \otimes \mathcal{O}(\mathbb{C}) + z^{-n/2+1}\mathscr{F}_\tau^{n-1} \otimes \mathcal{O}(\mathbb{C}) + \cdots + z^{n/2}\mathscr{F}_\tau^0 \otimes \mathcal{O}(\mathbb{C}).$$

The integral structure on the A-model $\frac{\infty}{2}$VHS does not induce a full integral lattice in H_0. One can see however that the lattice $\mathcal{V}_{\mathbb{Z},\rho}$ is naturally contained in H_0 by $A \mapsto z^{-\mu}A$ as a *partial* lattice. An integral period for $A \in \mathcal{V}_{\mathbb{Z},\rho}$ is related to a period of the finite-dimensional VHS $\{\mathscr{F}_\tau^p \subset H_0\}$ as follows.

Lemma 5.3. *For $A \in \mathcal{V}_{\mathbb{Z},\rho}$ and $\alpha \in H_{\mathrm{orb}}^{2n-2p}(\mathcal{X})$, the integral period $\Pi_A^\alpha(\tau, z)$ in* (84) *converges to the period of the finite-dimensional VHS $\mathscr{F}_\tau^p \subset H_0$ in the conformal limit:*

$$\lim_{\Re(s)\to\infty} z^{p-n/2}\Pi_A^\alpha(\tau - s\rho, z) = \left(z^{p-n/2}\mathbb{J}_\tau^{\mathrm{c}}\alpha, z^{-\mu}A\right)_{H_0} \in \mathbb{C}. \tag{87}$$

Note that $z^{p-n/2}\mathbb{J}_\tau^{\mathrm{c}}\alpha \in \mathscr{F}_\tau^p$ and that the limit depends only on $\tau \in H_{\mathrm{orb}}^2(\mathcal{X})/\mathbb{C}\rho$.

Remark 5.4. When the real structure $\mathcal{S}(\mathcal{X})_{\mathbb{Z}} \otimes \mathbb{R}$ makes $\mathbb{F}_\tau^{\mathrm{c}}$ a pure and polarized $\frac{\infty}{2}$VHS (see [49, Section 2]), the finite-dimensional VHS $\mathscr{F}_\tau^p$ satisfies the Hodge decomposition and Hodge–Riemann bilinear inequality:

$$H_0 = \mathscr{F}_\tau^p \oplus \kappa_{H_0}\left(\mathscr{F}_\tau^{n-p+1}\right), \qquad (-\mathbf{i})^{2p-n}\left(\phi, \kappa_{H_0}(\phi)\right)_{H_0} > 0$$

where κ_{H_0} is the real involution on H_0 and $\phi \in \mathscr{F}_\tau^p \cap \kappa_{H_0}(\mathscr{F}_\tau^{n-p})$.

5.3. Co-ordinates on $H_{\mathrm{orb}}^2(\mathcal{X})$ via integral periods

We use periods for $\mathscr{F}_\tau^n \subset H_0$ to construct a co-ordinate system on $H_{\mathrm{orb}}^2(\mathcal{X})$. Note that $\mathscr{F}_\tau^n = z^{n/2}\widehat{\mathbb{F}}^{\mathrm{c}} \cap H_0 = z^{n/2}\mathbb{J}_\tau^{\mathrm{c}}(H_{\mathrm{orb}}^0(\mathcal{X}))$ is one-dimensional over $\mathbb{C}$. Using the Galois action, we take a good set of integral vectors in $\mathcal{V}_{\mathbb{Z},\rho}$ to measure $\mathscr{F}_\tau^n$.

Choose an ample line bundle L pulled back from the coarse moduli space X of $\mathcal{X}$. Then the Galois action $G^{\mathcal{S}}([L])$ is unipotent (since $f_v([L]) = 0$ in (20)) and its logarithm $\mathcal{N} = \mathrm{Log}(\mathcal{Z}_{\mathrm{coh}}^{-1}G^{\mathcal{S}}([L])\mathcal{Z}_{\mathrm{coh}}) = -2\pi\mathbf{i}c_1(L)$ defines a weight filtration W_k on $\mathcal{V}$. This is an increasing filtration characterized by the condition:

$$\mathcal{N}W_k \subset W_{k-2}, \qquad \mathcal{N}^k : \mathrm{Gr}_k^W(\mathcal{V}) \cong \mathrm{Gr}_{-k}^W(\mathcal{V}),$$

where $\mathrm{Gr}_k^W(\mathcal{V}) = W_k/W_{k-1}$. It is given by (independent of a choice of L)

$$W_k = \bigoplus_{v\in\mathsf{T}} H^{\geqslant n_v - k}(\mathcal{X}_v). \tag{88}$$

The weight filtration is defined over $\mathbb{Q}$ (with respect to the lattice $\mathcal{V}_{\mathbb{Z}}$). Similarly, the subspace $\mathrm{Ker}(H^2(\mathcal{X})) := \{\alpha \in \mathcal{V} = H^*_{\mathrm{orb}}(\mathcal{X});\ \tau_{0,2} \cdot \alpha = 0,\ \forall \tau_{0,2} \in H^2(\mathcal{X})\}$ is also characterized by Galois actions and is defined over $\mathbb{Q}$. These subspaces define the following filtration on $\mathcal{V}_{\mathbb{Z},\rho}$:

$$(W_{-n} \cap \mathcal{V}_{\mathbb{Z},\rho}) \subset \big(\mathrm{Ker}\big(H^2(\mathcal{X})\big) \cap W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho}\big) \subset (W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho})$$

which are full lattices of the vector spaces:

$$H^{2n}(\mathcal{X}) \subset H^{2n}(\mathcal{X}) \oplus \bigoplus_{n_v=n-2} H^{2n_v}(\mathcal{X}_v) \subset \big(H^{\geqslant 2n-2}(\mathcal{X}) \cap \mathrm{Ker}(\rho)\big) \oplus \bigoplus_{n_v=n-2} H^{2n_v}(\mathcal{X}_v).$$

Since $\mathcal{X}$ is Gorenstein, we have no $v \in \mathsf{T}$ satisfying $n_v = n-1$ and $\iota_v = 1$ if $n_v = n-2$. Thus these subspaces are contained in $H^{\geqslant 2n-2}_{\mathrm{orb}}(\mathcal{X})$. We take integral vectors $A_0, A_1, \ldots, A_\flat, A_{\flat+1}, \ldots, A_\sharp$ in $\mathcal{V}_{\mathbb{Z},\rho}$ compatible with this filtration:

$$W_{-n} \cap \mathcal{V}_{\mathbb{Z},\rho} = \mathbb{Z}A_0,$$

$$\mathrm{Ker}\big(H^2(\mathcal{X})\big) \cap W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho} = \mathbb{Z}A_0 + \sum_{i=1}^{\flat} \mathbb{Z}A_i,$$

$$W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho} = \mathbb{Z}A_0 + \sum_{i=1}^{\flat} \mathbb{Z}A_i + \sum_{i=\flat+1}^{\sharp} \mathbb{Z}A_i.$$

The vector $A_0 \in H^{2n}(\mathcal{X})$ is unique up to sign and invariant under all Galois action. In analogy with the Calabi–Yau B-model, we normalize a generator $\Omega_\tau \in \mathscr{F}^n_\tau = z^{n/2}\mathbb{J}^{\mathrm{c}}(H^0_{\mathrm{orb}}(\mathcal{X}))$ by the condition

$$\big(\Omega_\tau, z^{-\mu} A_0\big)_{H_0} = 1. \tag{89}$$

Using the expression (86), one can easily see that $\Omega_\tau = z^{n/2}\mathbb{J}^{\mathrm{c}}_\tau((\mathbf{i}^n a_0)^{-1}\mathbf{1})$ with $a_0 := (\mathbf{1}, A_0)_{\mathrm{orb}}$. The *normalized integral period* $\overline{\Pi}_A(\tau)$ of $A \in \mathcal{V}_{\mathbb{Z},\rho}$ is defined by (cf. (87))

$$\overline{\Pi}_A(\tau) := \big(\Omega_\tau, z^{-\mu} A\big)_{H_0}, \quad \tau \in H^2_{\mathrm{orb}}(\mathcal{X}).$$

The filter $W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho}$ does not necessarily span $H^{\geqslant 2n-2}_{\mathrm{orb}}(\mathcal{X}) \cap \mathrm{Ker}(\rho)$.

Proposition 5.5. *For $\tau \in H^2_{\mathrm{orb}}(\mathcal{X})$, we write $\tau = \tau_{0,2} + \tau_{\mathrm{tw}} = \tau_{0,2} + \tau'_{\mathrm{tw}} + \tau''_{\mathrm{tw}}$ with $\tau_{0,2} \in H^2(\mathcal{X})$, $\tau_{\mathrm{tw}} \in \bigoplus_{\iota_v=1} H^0(\mathcal{X}_v)$, $\tau'_{\mathrm{tw}} \in \bigoplus_{n_v=n-2} H^0(\mathcal{X}_v)$ and $\tau''_{\mathrm{tw}} \in \bigoplus_{n_v<n-2,\ \iota_v=1} H^0(\mathcal{X}_v)$. Set $a_i := (\mathbf{1}, A_i)_{\mathrm{orb}}$. The normalized integral periods $\overline{\Pi}_{A_i}(\tau)$ give an affine co-ordinate system on the space $(H^2(\mathcal{X})/\mathbb{C}\rho) \oplus \bigoplus_{n_v=n-2} H^0(\mathcal{X}_v)$:*

$$\overline{\Pi}_{A_i}(\tau) = a_0^{-1} a_i - \big(\tau'_{\mathrm{tw}}, a_0^{-1} A_i\big)_{\mathrm{orb}}, \quad 1 \leqslant i \leqslant \flat,$$

$$\overline{\Pi}_{A_i}(\tau) = a_0^{-1} a_i - \big(\tau'_{\mathrm{tw}}, a_0^{-1} A_i\big)_{\mathrm{orb}} - \frac{1}{2\pi\mathbf{i}} \tau_{0,2} \cap [C_i], \quad \flat + 1 \leqslant i \leqslant \sharp,$$

where $[C_i] \in H_2(\mathcal{X})$ *is the Poincaré dual of the* $H^{2n-2}(\mathcal{X})$*-component of* $2\pi\mathbf{i}a_0^{-1}A_i$ *and*

$$[C_i] \in H_2(X,\mathbb{Z}) \cap \operatorname{Ker}\rho, \quad \textit{where } X \textit{ is the coarse moduli space of } \mathcal{X}. \tag{90}$$

Here, $[C_{\flat+1}],\dots,[C_\sharp]$ *form a* $\mathbb{Q}$*-basis of* $H_2(X,\mathbb{Q}) \cap \operatorname{Ker}\rho$*. The period of a class* $B \in \operatorname{Ker}(H^2(\mathcal{X})) \cap \mathcal{V}_{\mathbb{Z},\rho}$ *is possibly non-linear and has the asymptotic*

$$\overline{\Pi}_B(\tau) \sim a_0^{-1}b - \left(\tau_{\mathrm{tw}}, a_0^{-1}B\right)_{\mathrm{orb}}, \quad b := (\mathbf{1}, B)_{\mathrm{orb}},$$

in the large radius limit (5)*. The constant term* $a_0^{-1}a_i$ *(resp.* $a_0^{-1}b$*) is a rational number if the following condition* (91) *(resp.* (92)*) holds:*

$$\textit{The projection } W_{-n+2} \cap \operatorname{Ker}(\rho) \to H^{2n}(\mathcal{X}) \textit{ is defined over } \mathbb{Q}, \tag{91}$$

$$\begin{cases} H^*(\mathcal{X}) \textit{ is generated by } H^2(\mathcal{X}) \textit{ and} \\ \forall v \in \mathsf{T}\ \left(v \neq 0 \implies \exists \xi \in H^2(\mathcal{X},\mathbb{Z}) \textit{ such that } f_v(\xi) > 0\right). \end{cases} \tag{92}$$

Here the projection in (91) *is to take the* $H^{2n}(\mathcal{X})$*-component. Recall that* $W_{-n+2} \cap \operatorname{Ker}(\rho) = (H^{\geqslant 2n-2}(\mathcal{X}) \cap \operatorname{Ker}(\rho)) \oplus \bigoplus_{n_v=n-2} H^{2n_v}(\mathcal{X}_v)$.

Proof. By (86) and the string equation (see [2]), $\mathbb{J}^{\mathrm{c}}_\tau\mathbf{1}$ can be written as follows:

$$\begin{aligned} \mathbb{J}^{\mathrm{c}}_\tau\mathbf{1} &= e^{\tau_{0,2}/z}\left(1 + \frac{\tau_{\mathrm{tw}}}{z} + \sum_{\substack{d \in \mathrm{Eff}_{\mathcal{X}} \cap \operatorname{Ker}(\rho) \\ l \geqslant 0 \\ d=0 \Rightarrow l \geqslant 2}} \sum_{i=1}^{N} \left\langle \tau_{\mathrm{tw}}, \dots, \tau_{\mathrm{tw}}, \frac{\phi_i}{z(z-\psi)} \right\rangle_{0,l+1d}^{\mathcal{X}} e^{\langle \tau_{0,2}, d\rangle} \phi^i \right) \\ &= 1 + \frac{\tau}{z} + z^{-2} H^{\geqslant 4}_{\mathrm{orb}}(\mathcal{X}) \otimes \mathbb{C}\left[z^{-1}\right]. \end{aligned}$$

The expressions for $\overline{\Pi}_{A_i}(\tau)$, $\overline{\Pi}_B(\tau)$ easily follow from this.

If $\xi \in H^2(X,\mathbb{Z})$ is an integral class on the coarse moduli space, $G^{\mathcal{S}}(\xi)$ acts on $\mathcal{V}$ by $e^{-2\pi\mathbf{i}\xi}$ by (20). Because the Galois action preserves the integral structure, $e^{-2\pi\mathbf{i}\xi}A_i = A_i - m_iA_0$ for some integer m_i. Here, $2\pi\mathbf{i}\xi A_i = m_iA_0$. Hence, $\xi \cap [C_i] = (\xi, 2\pi\mathbf{i}a_0^{-1}A_i)_{\mathrm{orb}} = a_0^{-1}(\mathbf{1}, 2\pi\mathbf{i}\xi A_i)_{\mathrm{orb}} = m_i \in \mathbb{Z}$. This shows (90).

Under the condition (91), the $H^{2n}(\mathcal{X})$-component of A_i is of the form c_iA_0 for $c_i \in \mathbb{Q}$. Hence $a_i = (\mathbf{1}, A_i)_{\mathrm{orb}} = c_i(\mathbf{1}, A_0) = c_ia_0$ and $a_0^{-1}a_i$ is rational.

Under the condition (92), we have the decomposition $\operatorname{Ker}(H^2(\mathcal{X})) = H^{2n}(\mathcal{X}) \oplus (\operatorname{Ker}(H^2(\mathcal{X})) \cap \bigoplus_{v\in\mathsf{T}} H^*(\mathcal{X}_v))$. By a consideration of the Galois action, we can easily see that this is defined over $\mathbb{Q}$. The rationality of $a_0^{-1}b$ follows similarly. □

Remark 5.6. The rationality of $a_0^{-1}a_i$, $a_0^{-1}b$ are related to the rationality of specialization values in crepant resolution conjecture. The condition (91) is satisfied by the $\widehat{\Gamma}$-integral structure. See Section 5.4 below.

5.4. Example: $\widehat{\Gamma}$-integral structure

Here we take $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ to be the $\widehat{\Gamma}$-integral structure in Definition 2.9 and compute some examples of integral periods. The lattice $\mathcal{V}_{\mathbb{Z}}$ is given by $\Psi(K(\mathcal{X}))$. By a natural map from the K-group of coherent sheaves to the K-group of topological orbifold vector bundles, we can regard a coherent sheaf as an element of $K(\mathcal{X})$. The integral vector $A_0 \in W_{-n} \cap \mathcal{V}_{\mathbb{Z},\rho}$ comes from the structure sheaf $\mathcal{O}_x$ of a non-stacky point $x \in \mathcal{X}$:

$$A_0 = \Psi(\mathcal{O}_x) = \frac{(2\pi\mathbf{i})^n}{(2\pi)^{n/2}}[\mathrm{pt}].$$

Here, we used the Poincaré duality to identify $[\mathrm{pt}] \in H_0(\mathcal{X})$ with an element in $H^{2n}(\mathcal{X})$. Hence we have $\Omega_\tau = (-1)^n(2\pi)^{-n/2} z^{n/2} \mathbb{J}_\tau^{\mathrm{c}} \mathbf{1}$.

5.4.1. A smooth curve

Let $\mathcal{X} = X$ be a manifold and $C \subset X$ be a smooth curve of genus g such that $[C] \cap c_1(\mathcal{X}) = 0$. Then the structure sheaf $\mathcal{O}_C(g-1)$ defines an integral vector $A_C \in W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho}$

$$A_C := \Psi\big(\mathcal{O}_C(g-1)\big) = \frac{(2\pi\mathbf{i})^{n-1}}{(2\pi)^{n/2}}[C]$$

and an integral period

$$\overline{\Pi}_{A_C}(\tau) = -\frac{1}{2\pi\mathbf{i}}[C] \cap \tau.$$

5.4.2. A general element in $W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho}$

Let $\Psi(V) \in W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho}$ be an arbitrary element. Using the fact that the untwisted sector of $\widehat{\Gamma}(T\mathcal{X})$ is of the form $1 - \gamma\rho + H^{\geqslant 4}(\mathcal{X})$ (γ is the Euler constant) and that $\rho \cdot \widetilde{\mathrm{ch}}(V) = 0$, we can see that the $H^{2n}(\mathcal{X})$ component of $\Psi(V)$ belongs to $(2\pi)^{-n/2}(2\pi\mathbf{i})^n H^{2n}(\mathcal{X},\mathbb{Q}) = \mathbb{Q}A_0$. Therefore, the condition (91) holds for the $\widehat{\Gamma}$-integral structure. We have

$$\overline{\Pi}_{\Psi(V)}(\tau) = \int_{\mathcal{X}} \mathrm{ch}(V) - \big(\tau'_{\mathrm{tw}}, a_0^{-1}\Psi(V)\big)_{\mathrm{orb}} - \frac{1}{2\pi\mathbf{i}}\tau_{0,2} \cap [C]$$

for some $[C] \in H_2(X,\mathbb{Z}) \cap \operatorname{Ker}\rho$ and $a_0 = (2\pi)^{-n/2}(2\pi\mathbf{i})^n$.

5.4.3. A stacky point

Let $y \in \mathcal{X}$ be a possibly stacky point. Let $\varrho : \mathrm{Aut}(y) \to \mathrm{End}(V)$ be a finite-dimensional representation of the automorphism group of y. This defines a coherent sheaf $\mathcal{O}_y \otimes V$ supported on y and an integral vector $A_{(y,V)} := \Psi(\mathcal{O}_y \otimes V) \in \mathrm{Ker}(H^2(\mathcal{X})) \cap \mathcal{V}_{\mathbb{Z},\rho}$. Using Toën's Riemann–Roch theorem [70], one calculates

$$A_{(y,V)} = \frac{(2\pi\mathbf{i})^n}{(2\pi)^{n/2}} \sum_{(g)\subset \mathrm{Aut}(y)} \frac{(-1)^{n+n_{v(g)}+\iota_{v(g)}} \operatorname{Tr}(\varrho(g^{-1}))}{|C(g)| \prod_{j=1}^{n-n_{v(g)}} \Gamma(f_{g,j})}[\mathrm{pt}]_{v(g)},$$

where the sum is over all conjugacy classes (g) of $g \in \mathrm{Aut}(y)$, $C(g)$ is the centralizer of g, $v(g) \in \mathsf{T}$ is the inertia component containing $(y,g) \in I\mathcal{X}$, $[\mathrm{pt}]_{v(g)}$ is the homology class of a point on $\mathcal{X}_{v(g)}$ (represented by a map $\mathrm{pt} \to \mathcal{X}_v$ of stacks), $f_{g,1}, \ldots, f_{g,n-n_{v(g)}}$ are rational numbers in $(0,1)$ such that $\{e^{2\pi \mathbf{i} f_{g,j}}\}_j$ is a multi-set of the eigenvalues $\neq 1$ of the g action on $T_y\mathcal{X}$. The corresponding integral period behaves

$$\overline{\Pi}_{A_{(y,V)}}(\tau) \sim \frac{\dim(V)}{|\mathrm{Aut}(y)|} + \sum_{\substack{(g)\subset \mathrm{Aut}(y)\\ \iota_{v(g)}=1}} \frac{\mathrm{Tr}(\varrho(g))}{|C(g)|\prod_{j=1}^{n-n_{v(g)}} \Gamma(1-f_{g,j})} \tau_{\mathrm{tw}} \cap [\mathrm{pt}]_{v(g)}$$

in the large radius limit. This is an exact formula if $y \notin \mathcal{X}_v$ for all v with $\operatorname{codim} \mathcal{X}_v = n - n_v \geqslant 3$ or equivalently, $A_{(y,V)} \in \mathrm{Ker}(H^2(\mathcal{X})) \cap W_{-n+2} \cap \mathcal{V}_{\mathbb{Z},\rho}$.

Note that the subspace $\mathcal{V}_{\mathrm{top}} := \bigoplus_{v\in\mathsf{T}} H^{2n_v}(\mathcal{X}_v) \subset \mathcal{V}$ is spanned by the integral vectors $A_{(y,V)}$ above, so is defined over $\mathbb{Q}$ for the $\widehat{\Gamma}$-integral structure. (This may not be true for an arbitrary integral structure.) For an integral vector $\Psi(V)$ in $\mathcal{V}_{\mathrm{top}}$, the period $\overline{\Pi}_{\Psi(V)}(\tau)$ takes the rational value $\int_{\mathcal{X}} \mathrm{ch}(V)$ at the large radius limit.

5.5. *Crepant resolution conjecture with an integral structure*

Yongbin Ruan's crepant resolution conjecture [66] states that when Y is a crepant resolution of the coarse moduli space X of a Gorenstein orbifold $\mathcal{X}$,

$$\pi : Y \to X, \quad \pi^*(K_X) = K_Y,$$

the (orbifold) quantum cohomology of $\mathcal{X}$ and Y are related by analytic continuation in quantum parameters. This conjecture was formulated more precisely by Bryan and Graber [15] as an isomorphism of Frobenius manifolds (under the Hard Lefschetz condition). In the joint work [27] with Coates and Tseng, based on the toric mirror picture, we gave a conjecture that the A-model $\frac{\infty}{2}$VHS of $\mathcal{X}$ and Y are related by an $\mathcal{O}(\mathbb{C}^*)$-linear symplectic transformation $\mathbb{U}: \mathcal{H}^{\mathcal{X}} \to \mathcal{H}^{Y}$ between the Givental spaces. (This does not need the Hard Lefschetz condition.) This symplectic transformation $\mathbb{U}$ encodes all the information on relationships between the genus zero Gromov–Witten theories of $\mathcal{X}$ and Y. See [23,50] for expositions and [21] for local examples.

In this section, we incorporate integral structures into this picture and propose a possible relationship between the K-group McKay correspondence and the crepant resolution conjecture. We use a superscript to distinguish the spaces $\mathcal{X}$, Y, e.g. $\mathcal{H}^{\mathcal{X}}$, $\mathcal{H}^{Y}$, etc.

Proposal 5.7.

(a) For each smooth Deligne–Mumford stack $\mathcal{X}$ with a projective coarse moduli space, the space $\mathcal{S}(\mathcal{X})$ of flat sections of the quantum D-module admits a $\mathbb{Z}$-lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ which is given by the image of the topological K-group under a *K-group framing* $\mathcal{Z}_K^{\mathcal{X}}$:

$$\mathcal{Z}_K^{\mathcal{X}} : K(\mathcal{X}) \to \mathcal{S}(\mathcal{X}), \qquad V \mapsto L(\tau,z) z^{-\mu} z^{\rho} \Psi^{\mathcal{X}}(V),$$

where $\Psi^{\mathcal{X}}$ is a map from $K(\mathcal{X})$ to $H^*_{\mathrm{orb}}(\mathcal{X})$ and $L(\tau,z)$ is the fundamental solution (11). We hope that $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ is given by the $\widehat{\Gamma}$-integral structure, namely, $\Psi^{\mathcal{X}}$ is given by (24). In the discussion below, we only need to assume that $\mathcal{Z}_K^{\mathcal{X}}$ satisfies the conclusions of Proposition 2.10.

(b) Let Y be a crepant resolution of the coarse moduli space X of a Gorenstein orbifold $\mathcal{X}$. The *K-group McKay correspondence* predicts that we have an isomorphism of K-groups

$$\mathbb{U}_K : K(\mathcal{X}) \cong K(Y)$$

which preserves the Mukai pairing (given in Proposition 2.10) and commutes with the tensor by a topological line bundle L on the coarse moduli space of $\mathcal{X}$, $\mathbb{U}_K(L \otimes \cdot) = \pi^*(L) \otimes \mathbb{U}_K(\cdot)$.

(c) The quantum D-modules $QDM(\mathcal{X})$, $QDM(Y)$ with integral structures $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$, $\mathcal{S}(Y)_{\mathbb{Z}}$ become isomorphic under analytic continuation. The isomorphism of $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ and $\mathcal{S}(Y)_{\mathbb{Z}}$ are induced from the K-group McKay correspondence $\mathbb{U}_K : K(\mathcal{X}) \to K(Y)$ via the K-group framings.

In terms of the $\frac{\infty}{2}$VHS introduced in Section 2.5, we have a degree-preserving[11] $\mathcal{O}(\mathbb{C}^*)$-linear symplectic isomorphism $\mathbb{U} : \mathcal{H}^{\mathcal{X}} \to \mathcal{H}^Y$ and a map Υ from a subdomain of $H^*_{\mathrm{orb}}(\mathcal{X})$ to a subdomain of $H^*(Y)$ (where the quantum cohomology is analytically continued) such that the $\frac{\infty}{2}$VHS of $\mathcal{X}$ and Y are identified by $\mathbb{U}$

$$\mathbb{U}\big(\mathbb{F}_\tau^{\mathcal{X}}\big) = \mathbb{F}_{\Upsilon(\tau)}^Y$$

and that $\mathbb{U}$ is induced from $\mathbb{U}_K$ by the commutative diagram:

$$\begin{tikzcd} K(\mathcal{X}) \arrow[r, "\mathbb{U}_K"] \arrow[d, "z^{-\mu} z^{\rho} \Psi^{\mathcal{X}}"'] & K(Y) \arrow[d, "z^{-\mu} z^{\rho} \Psi^{Y}"] \\ \mathcal{H}^{\mathcal{X}} \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}(\widetilde{\mathbb{C}^*}) \arrow[r, "\mathbb{U}"] & \mathcal{H}^{Y} \otimes_{\mathcal{O}(\mathbb{C}^*)} \mathcal{O}(\widetilde{\mathbb{C}^*}) \end{tikzcd} \tag{93}$$

where μ, ρ in the left/right vertical arrow are those for $\mathcal{X}/Y$.

We hope that the isomorphism $\mathbb{U}_K$ in (b) arises from a geometric correspondence such as Fourier–Mukai transformations. In fact, Borisov and Horja [11] showed that an analytic continuation of solutions to the GKZ-system corresponds to a Fourier–Mukai transformation between K-groups of toric Calabi–Yau orbifolds.

Remark 5.8. As formulated in [23,27], the symplectic transformation $\mathbb{U}$ identifies the Givental's Lagrangian cone (30), i.e. $\mathbb{U}\mathcal{L}^{\mathcal{X}} = \mathcal{L}^Y$. Thus the relationship of the genus zero descendant potentials of $\mathcal{X}$ and Y is completely described by $\mathbb{U}$.

We discuss what follows from this proposal assuming $\mathcal{X}$ is weak Fano, i.e. $c_1(\mathcal{X})$ is nef. As discussed in [27], this picture implies that quantum cohomology of $\mathcal{X}$ and Y are identified via Υ and $\mathbb{U}$ as a family of algebras (not necessarily as Frobenius manifolds). However, the large radius limit points for $\mathcal{X}$ and Y are not identified under Υ, so we need actual analytic continuations. We

[11] The grading on $\mathcal{H}$ is given by $\deg z = 2$ and the grading on orbifold cohomology.

refer the reader to [23,27,50] for these things. Let us first observe that integral periods of $\mathcal{X}$ and Y in the conformal limit match under Υ and $\mathbb{U}$ (see (95) below). Because $\mathbb{U}_K$ commutes with the tensor by a line bundle pulled back from X, it follows that $\mathbb{U}$ must commute with $H^2(\mathcal{X})$ ((b), Section 5 in [27]; (b), Conjecture 4.1 in [23]), i.e.

$$\mathbb{U}(\alpha \cup \cdot) = \pi^*(\alpha) \cup \mathbb{U}(\cdot), \quad \alpha \in H^2(\mathcal{X}). \tag{94}$$

Since $\mathcal{X}$ is weak Fano, by the discussion leading to Theorem 8.2 in [23] (essentially using Lemma 5.1 in [23]), we know that Υ should map $H^2_{\mathrm{orb}}(\mathcal{X})$ to $H^2(Y)$:

$$\Upsilon\big(H^2_{\mathrm{orb}}(\mathcal{X})\big) \subset H^2(Y).$$

The conformal limit $\tau \to \tau - s\rho$, $\Re(s) \to \infty$ on $H^2_{\mathrm{orb}}(\mathcal{X})$ should also be mapped to the conformal limit on $H^2(Y)$ under Υ because this flow is generated by the Euler vector field and the two Euler vector fields should match under Υ (the Euler vector field is a part of the data of a quantum D-module). Therefore, by (94) and $\pi^* c_1(\mathcal{X}) = c_1(Y)$, the conformal limit of the $\frac{\infty}{2}$VHSs (Definition 5.1) also match under $\mathbb{U}$:

$$\mathbb{U}\big(\mathbb{F}^{\mathrm{c},\mathcal{X}}_\tau\big) = \mathbb{F}^{\mathrm{c},Y}_{\Upsilon(\tau)}.$$

In particular, the finite-dimensional VHSs $(\mathscr{F}^{\mathcal{X},\bullet}_\tau \subset H^{\mathcal{X}}_0)$, $(\mathscr{F}^{Y,\bullet}_\tau \subset H^Y_0)$ associated with these also match:

$$\mathbb{U}\big(\mathscr{F}^{\mathcal{X},\bullet}_\tau\big) = \mathscr{F}^{Y,\bullet}_{\Upsilon(\tau)}, \quad \mathbb{U}: \widehat{\mathcal{H}}^{\mathcal{X}} \supset H^{\mathcal{X}}_0 \to H^Y_0 \subset \widehat{\mathcal{H}}^Y.$$

We used the fact that $\mathbb{U}$ induces a map from $H^{\mathcal{X}}_0 = \mathrm{Ker}(z\partial_z + \mu^{\mathcal{X}})$ to $H^Y_0 = \mathrm{Ker}(z\partial_z + \mu^Y)$. Set $\mathcal{V}^{\mathcal{X}} := H^*_{\mathrm{orb}}(\mathcal{X})$, $\mathcal{V}^Y := H^*(Y)$ and let $\mathbb{U}_{\mathcal{V}} : \mathcal{V}^{\mathcal{X}} \to \mathcal{V}^Y$ be the map induced from $\mathbb{U}_K$ (via Ψ)

$$\begin{tikzcd} K(\mathcal{X}) \arrow[r, "\mathbb{U}_K"] \arrow[d, "\Psi^{\mathcal{X}}"'] & K(Y) \arrow[d, "\Psi^Y"] \\ \mathcal{V}^{\mathcal{X}} \arrow[r, "\mathbb{U}_{\mathcal{V}}"] & \mathcal{V}^Y. \end{tikzcd}$$

This again commutes with $H^2(\mathcal{X})$ and is related to $\mathbb{U}$ by

$$\mathbb{U} = z^{-\mu^Y} z^{\rho^Y} \mathbb{U}_{\mathcal{V}} z^{-\rho^{\mathcal{X}}} z^{\mu^{\mathcal{X}}} = z^{-\mu^Y} \mathbb{U}_{\mathcal{V}} z^{\mu^{\mathcal{X}}}.$$

The integral structures $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$, $\mathcal{S}(Y)_{\mathbb{Z}}$ induce the lattices $\mathcal{V}^{\mathcal{X}}_{\mathbb{Z}} = \mathcal{Z}^{-1}_{\mathrm{coh}}(\mathcal{S}(\mathcal{X})_{\mathbb{Z}}) = \Psi^{\mathcal{X}}(K(\mathcal{X}))$, $\mathcal{V}^Y_{\mathbb{Z}} = \mathcal{Z}^{-1}_{\mathrm{coh}}(\mathcal{S}(Y)_{\mathbb{Z}}) = \Psi^Y(K(Y))$ as before. Let L be an ample line bundle on X. Consider the weight filtration $W^{\mathcal{X}}_k$ (88) on $\mathcal{V}^{\mathcal{X}}$ defined by the Galois action logarithm $-2\pi\mathbf{i}c_1(L)$. The first term $W^{\mathcal{X}}_{-n}$ of the weight filtration is given by $\mathrm{Im}(c_1(L)^n)$. Thus $\mathbb{U}_{\mathcal{V}}(W^{\mathcal{X}}_{-n}) = \mathrm{Im}(\pi^*(c_1(L))^n) = H^{2n}(Y)$. Note that $\pi^*(c_1(L))^n$ is non-trivial since $\pi : Y \to X$ is birational. Therefore, for the weight filtration W^Y_k on $\mathcal{V}^Y$ (defined by an ample class on Y), we have

$$\mathbb{U}_{\mathcal{V}}\big(W^{\mathcal{X}}_{-n}\big) = W^Y_{-n}.$$

As we did before, we use an integral vector $A_0^{\mathcal{X}}$ (unique up to sign) in $W_{-n}^{\mathcal{X}} \cap \mathcal{V}_{\mathbb{Z},\rho}^{\mathcal{X}}$ to normalize a generator $\Omega_\tau^{\mathcal{X}} \in \mathscr{F}_\tau^{\mathcal{X},n}$ and then use $A_0^Y := \mathbb{U}_{\mathcal{V}}(A_0^{\mathcal{X}}) \in W_{-n}^Y \cap \mathcal{V}_{\mathbb{Z},\rho}^Y$ to normalize $\Omega_\tau^Y \in \mathscr{F}_\tau^{Y,n}$ (see (89)). Because the $\mathbb{U}$ preserves the pairing, we have

$$\mathbb{U}\big(\Omega_\tau^{\mathcal{X}}\big) = \Omega_{\Upsilon(\tau)}^Y.$$

When $A^{\mathcal{X}} \in \mathcal{V}_{\mathbb{Z},\rho}^{\mathcal{X}} = \mathcal{V}_{\mathbb{Z}}^{\mathcal{X}} \cap \operatorname{Ker}(c_1(\mathcal{X}))$, the corresponding vector $A^Y = \mathbb{U}_{\mathcal{V}}(A^{\mathcal{X}})$ belongs to $\mathcal{V}_{\mathbb{Z}}^Y \cap \operatorname{Ker}(\pi^*(c_1(\mathcal{X}))) = \mathcal{V}_{\mathbb{Z},\rho}^Y$ and the integral periods match

$$\overline{\Pi}_{A^{\mathcal{X}}}^{\mathcal{X}}(\tau) = \big(\Omega_\tau^{\mathcal{X}}, z^{-\mu} A^{\mathcal{X}}\big)_{H_0^{\mathcal{X}}} = \big(\Omega_{\Upsilon(\tau)}^Y, z^{-\mu} A^Y\big)_{H_0^Y} = \overline{\Pi}_{A^Y}^Y\big(\Upsilon(\tau)\big). \tag{95}$$

Now we can make predictions on the specialization values of quantum parameters. Note that $\operatorname{Ker}(\pi^* H^2(\mathcal{X})) \subset \mathcal{V}^Y$ is defined over $\mathbb{Q}$. Take a basis $A_0^Y, A_1^Y, \dots, A_\natural^Y$ of $\operatorname{Ker}(\pi^* H^2(\mathcal{X})) \cap W_{-n+2}^Y \cap \mathcal{V}_{\mathbb{Z},\rho}^Y$. These generate a full lattice in $H^{2n}(Y) \oplus (H^{2n-2}(Y) \cap \operatorname{Ker}\pi_*)$ over $\mathbb{C}$. By Proposition 5.5, the integral periods for $A_1^Y, \dots, A_\natural^Y$ are of the form:

$$\overline{\Pi}_{A_i^Y}^Y(\tau) = a_0^{-1} a_i - \frac{1}{2\pi\mathbf{i}}[C_i] \cap \tau, \quad a_i := \big(A_i^Y, \mathbf{1}\big)_{\mathrm{orb}}. \tag{96}$$

Here $[C_1], \dots, [C_\natural] \in H_2(Y, \mathbb{Z}) \cap \operatorname{Ker}\pi_*$ are a $\mathbb{Q}$-basis of $H_2(Y, \mathbb{Q}) \cap \operatorname{Ker}\pi_*$. So $\overline{\Pi}_{A_i^Y}^Y(\tau)$, $1 \leqslant i \leqslant \natural$, form an affine co-ordinate system on $H^2(Y)/\operatorname{Im}\pi^*$. The integral vector $A_i^{\mathcal{X}}$ corresponding to A_i^Y belongs to $\operatorname{Ker}(H^2(\mathcal{X})) \cap \mathcal{V}_{\mathbb{Z},\rho}^{\mathcal{X}}$. From (95), Proposition 5.5 and examples in Section 5.4, Proposal 5.7 yields the following prediction:

(i) Assume that the condition (92) holds for $\mathcal{X}$. Then the integral periods $\overline{\Pi}_{A_i^Y}^Y(\tau)$ (96) for Y take rational values at the large radius limit point of $\mathcal{X}$.
(ii) Assume in addition to (i) that the condition (91) (with $\mathcal{X}$ there replaced with Y) holds for the rational structure on $\mathcal{V}^Y$. Then $a_0^{-1} a_i$ in (96) is rational, so the "quantum parameter" $q_C := \exp([C] \cap \tau)$ with $[C] \in H_2(Y, \mathbb{Z}) \cap \operatorname{Ker}\pi_*$ for Y specializes to a root of unity at the large radius limit point of $\mathcal{X}$.
(iii) Assume that Proposal 5.7 holds for the $\widehat{\Gamma}$-integral structures on $\mathcal{X}$ and Y. Let $C \subset Y$ be a smooth rational curve in the exceptional set. If $\mathbb{U}_K^{-1}$ sends $[\mathcal{O}_C(-1)] \in K(Y)$ to $[\mathcal{O}_x \otimes V] \in K(\mathcal{X})$ for $x = \pi(C)$ and some representation V of $\operatorname{Aut}(x)$, the quantum parameter q_C specializes to $\exp(-2\pi\mathbf{i}\dim V/|\operatorname{Aut}(x)|)$ at the large radius limit point of $\mathcal{X}$.

For the A_n surface singularity resolution, each irreducible curve in the exceptional set corresponds to a one-dimensional irreducible representation of $\mathbb{Z}/(n+1)\mathbb{Z}$ under the McKay correspondence. If we use this McKay correspondence as $\mathbb{U}_K$, the prediction of specialization values made in (iii) is true [24]. Also, under the McKay correspondence, (iii) gives the same prediction (up to complex conjugation) made by Bryan and Graber [15], Bryan and Gholampour [14] for the ADE surface singularities and $\mathbb{C}^3/G$ with a finite subgroup $G \subset SO(3)$.

The equality (95) of integral periods can also predict the co-ordinate change Υ. See [50, Example 2.16, Section 3.8] for local Calabi–Yau examples.

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Appendix A

A.1. *Proof of Lemma 3.8*

Let Δ be a face of $\widehat{S}$ ($0 \leqslant \dim \Delta \leqslant n-1$). Let $B_\Delta \subset (\mathbb{C}^*)^r$ be the discriminant locus of $W_{q,\Delta}(y)$, i.e. the set of points $q = (q_1, \ldots, q_r)$ such that $W_{q,\Delta}(y)$ has a critical point $y \in (\mathbb{C}^*)^r$. It suffices to show that the closure $\overline{B_\Delta}$ of B_Δ in $\mathbb{C}^r$ does not contain the origin. Suppose $0 \in \overline{B_\Delta}$. Then there exists a curve $\alpha : \operatorname{Spec} \mathbb{C}[\![T]\!] \to \overline{B_\Delta}$ such that $\alpha(0) = 0$ and α restricts to $\alpha : \operatorname{Spec} \mathbb{C}(\!(T)\!) \to B_\Delta$. We can find a critical point $y(T)$ of $W_{q=\alpha(T),\Delta}(y)$ defined over the field $\overline{\mathbb{C}(\!(T)\!)} = \bigcup_{k\in\mathbb{N}} \mathbb{C}(\!(T^{1/k})\!)$ of Puiseux series. We take the leading terms of the T-expansions:

$$\begin{aligned} \alpha_a(T) &= c_a T^{d_a} + \text{h.o.t.}, \quad c_a \neq 0,\ 1 \leqslant a \leqslant r, \\ y_i(T) &= s_i T^{f_i} + \text{h.o.t.}, \quad s_i \neq 0,\ 1 \leqslant i \leqslant n. \end{aligned}$$

Note that $d_a > 0$ since $\alpha(0) = 0$. Put $h_i := \sum_{a=1}^r \ell_{ia} d_a$. (See Section 3.2.1 for ℓ_{ia}.) We claim that the piecewise linear function $h : \boldsymbol{N} \otimes \mathbb{R} \to \mathbb{R}$ on the fan Σ defined by $h(b_i) = h_i$ for $1 \leqslant i \leqslant m'$ is strictly convex (with respect to Σ) and $h(b_j) < h_j$ for $m' < j \leqslant m$. Since $\sum_{a=1}^r d_a p_a \in \widetilde{C}_{\mathcal{X}}$, for each "anticone" $I \in \mathcal{A}$, there exist $k_i > 0$, $i \in I$ such that $\sum_{a=1}^r d_a p_a = \sum_{i\in I} k_i D_i$. Using $p_a = \sum_{i=1}^m D_i \ell_{ia}$ and the exact sequence dual to (36), we have a linear function $\varphi : \boldsymbol{N} \otimes \mathbb{R} \to \mathbb{R}$ such that $\varphi(b_i) = h_i - k_i$ for $i \in I$ and $\varphi(b_i) = h_i$ for $i \notin I$. Since φ is a linear function which coincides with h on the cone $\sum_{i\notin I} \mathbb{R}_{\geqslant 0} b_i$, the claim follows. Now consider the leading term of the critical point equation $dW_{\alpha(T),\Delta}(y) = 0$:

$$0 = \sum_{b_i \in \Delta} \alpha(T)^{\ell_i} y(T)^{b_i} b_i = \Big(\sum c^{\ell_i} s^{b_i} b_i \Big) T^g + \text{h.o.t.},$$

where g is the minimal exponent and the last summation is over $1 \leqslant i \leqslant m$ such that $h_i + \sum_{j=1}^n b_{ij} f_j = g$ and $b_i \in \Delta$. The above claim shows that the b_i's appearing in the leading term span a cone in Σ and are linearly independent. This is a contradiction.

A.2. *Proof of Lemma 3.11*

Let $B \subset \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^*$ be a compact set. We need to show that $B' = \{(q, z, y);\ (q, z) \in B,\ y \in Y_q,\ \|df_{q,z}(y)\| \leqslant \epsilon\}$ is compact. Assume that there exists a divergent sequence $\{(q_{(k)}, z_{(k)}, y_{(k)})\}_{k=0}^\infty$

in B', i.e. any subsequence of it does not converge. Take an arbitrary Hermitian norm $\|\cdot\|$ on $\boldsymbol{N}\otimes\mathbb{C}$. Note that we have

$$\left\|df_{q,z}(y)\right\| = \frac{1}{|z|}\left\|\sum_{i=1}^{m} q^{\ell_i} y^{b_i} b_i\right\|.$$

By passing to a subsequence and renumbering $b_1,\dots,b_m$, we can assume that $q_{(k)}$ and $z_{(k)}$ converge and that $|y_{(k)}^{b_1}| \geqslant |y_{(k)}^{b_2}| \geqslant \cdots \geqslant |y_{(k)}^{b_m}|$ for all k. Since 0 is in the interior of $\widehat{S}$, there exist $c_i > 0$ such that $\sum_{i=1}^m c_i b_i = 0$. Hence $\prod_{i=1}^m |y_{(k)}^{b_i}|^{c_i} = 1$. Because $y_{(k)}$ diverges, we must have $\lim_{k\to\infty} |y_{(k)}^{b_1}| = \infty$. Since $\|df_{q_{(k)},z_{(k)}}(y_{(k)})\|$ is bounded, we have

$$0 = \lim_{k\to\infty} \frac{|z_{(k)}|}{|y_{(k)}^{b_1}|}\left\|df_{q_{(k)},z_{(k)}}(y_{(k)})\right\| = \lim_{k\to\infty}\left\|\sum_{i=1}^{m} q_{(k)}^{\ell_i} y_{(k)}^{b_i-b_1} b_i\right\|.$$

Because $|y_{(k)}^{b_i-b_1}| \leqslant 1$, by passing to a subsequence again, we can assume that $y_{(k)}^{b_i-b_1}$ converges to $\alpha_i \neq 0$ for all $1 \leqslant i \leqslant l$ and $y_{(k)}^{b_i-b_1}$ goes to 0 for $i > l$. Then we have

$$0 = \sum_{i=1}^{l} \tilde{q}^{\ell_i}\alpha_i b_i, \quad \tilde{q} = \lim_{k\to\infty} q_{(k)} \in \mathcal{M}^{\mathrm{o}}.$$

Put $\xi_{(k),i} := \log y_{(k),i}$. By choosing a suitable branch of the logarithm, we can assume that $\lim_{k\to\infty}\langle \xi_{(k)}, b_i - b_1\rangle = \log\alpha_i$ for $1 \leqslant i \leqslant l$ and $\lim_{k\to\infty}\langle \Re(\xi_{(k)}), b_i - b_1\rangle = -\infty$ for $i > l$. Let V be the $\mathbb{C}$ subspace of $\boldsymbol{N}\otimes\mathbb{C}$ spanned by $b_i - b_1$ with $1 \leqslant i \leqslant l$. Take the orthogonal decomposition $\boldsymbol{N}\otimes\mathbb{C} \cong V \oplus V^{\perp}$ and write $\xi_{(k)} = \xi'_{(k)} + \xi''_{(k)}$, where $\xi'_{(k)} \in V$ and $\xi''_{(k)} \in V^{\perp}$. Then $\xi'_{(k)}$ converges to some $\xi' \in V$. Putting $\tilde{y}_i = \exp(\xi'_i)$, we have $\tilde{y}^{b_i-b_1} = \alpha_i$ for $1 \leqslant i \leqslant l$ and so

$$\sum_{i=1}^{l} \tilde{q}^{\ell_i}\tilde{y}^{b_i} b_i = \tilde{y}^{b_1}\left(\sum_{i=1}^{l} \tilde{q}^{\ell_i}\tilde{y}^{b_i-b_1} b_i\right) = 0. \tag{97}$$

On the other hand, for a sufficiently big k, $\langle \Re(\xi''_{(k)}), b_i - b_1\rangle = 0$ for $1 \leqslant i \leqslant l$ and $\langle \Re(\xi''_{(k)}), b_i - b_1\rangle < 0$ for $i > l$. This means that $b_1,\dots,b_l$ are on some face Δ of $\widehat{S}$. But Eq. (97) shows that $\tilde{y}$ is a critical point of $W_{\tilde{q},\Delta}$. This contradicts to the assumption that $W_{\tilde{q}}$ is non-degenerate at infinity.

A.3. The pairings match under mirror symmetry

We give a proof of $(\cdot,\cdot)_{\mathcal{R}^{(0)}} = (\tau\times\mathrm{id})^*(\cdot,\cdot)_F$ in Proposition 4.8. Firstly we show that $(\cdot,\cdot)_{\mathcal{R}^{(0)}}$ is a constant multiple of $(\tau\times\mathrm{id})^*(\cdot,\cdot)_F$. The argument here follows the line of [27, Proposition 3.6], where the case $\mathcal{X} = \mathbb{P}(1,1,1,3)$ was discussed. We work on the (pulled back) A-model D-module via the identification Mir. Let

$$(\cdot,\cdot)_{\mathrm{B},(\tau(q),z)} : F_{(\tau(q),-z)} \times F_{(\tau(q),z)} \to \mathbb{C}$$

be the pairing induced from the B-model pairing $(\cdot,\cdot)_{\mathcal{R}^{(0)}}$ via Mir. Via the fundamental solution (11), this induces a pairing $(\!(\cdot,\cdot)\!)_{\mathrm{B}}$ on the Givental space $\mathcal{H}$ (see (26) and (28)):

$$\big(\!\big(\alpha(z),\beta(z)\big)\!\big)_{\mathrm{B}} := \big(L\big(\tau(q),-z\big)\alpha(-z), L\big(\tau(q),z\big)\beta(z)\big)_{\mathrm{B},(\tau(q),z)}$$

for $\alpha(z),\beta(z)\in\mathcal{H}$. Since $(\cdot,\cdot)_{\mathrm{B}}$ is ∇-flat, $(\!(\alpha(z),\beta(z))\!)_{\mathrm{B}}$ is independent of q. By the discussion after Conjecture 4.3, the monodromy over $\mathcal{M}$ gives all the Galois actions of $H^2(\mathcal{X},\mathbb{Z})$. Since the B-model pairing is monodromy-invariant, we have

$$\big(\!\big(\alpha(z),\beta(z)\big)\!\big)_{\mathrm{B}} = \big(\!\big(G^{\mathcal{H}}(\xi)\alpha(z), G^{\mathcal{H}}(\xi)\beta(z)\big)\!\big)_{\mathrm{B}}. \tag{98}$$

Taking $\xi\in H^2(\mathcal{X},\mathbb{Z})$ to be classes pulled-back from the coarse moduli space X (so that $f_v(\xi)=0$ for $v\in\mathsf{T}$) and using (29), one can deduce

$$\big(\!\big(\tau_{0,2}\cdot\alpha(z),\beta(z)\big)\!\big)_{\mathrm{B}} = \big(\!\big(\alpha(z),\tau_{0,2}\cdot\beta(z)\big)\!\big)_{\mathrm{B}},\quad \tau_{0,2}\in H^2(\mathcal{X}). \tag{99}$$

By (98) and (99), one can see that the semisimple part $\bigoplus_{v\in\mathsf{T}} e^{2\pi\mathbf{i}f_v(\xi)}$ of $G^{\mathcal{H}}(\xi)$ also preserves $(\!(\cdot,\cdot)\!)_{\mathrm{B}}$. This implies that, for $\alpha\in H^*(\mathcal{X}_v)$, $\beta\in H^*(\mathcal{X}_{v'})$,

$$(\!(\alpha,\beta)\!)_{\mathrm{B}} = 0 \quad \text{if } v'\neq \mathrm{inv}(v). \tag{100}$$

Here we used the fact that $v'=\mathrm{inv}(v)$ if $f_v(\xi)+f_{v'}(\xi)\in\mathbb{Z}$ for all $\xi\in H^2(\mathcal{X},\mathbb{Z})$. By the definition of $(\!(\cdot,\cdot)\!)_{\mathrm{B}}$, one has for $\alpha,\beta\in H^*_{\mathrm{orb}}(\mathcal{X})$,

$$\begin{aligned}(\alpha,\beta)_{\mathrm{B},(\tau(q),z)} &= \big(\!\big(L\big(\tau(q),z\big)^{-1}\alpha, L\big(\tau(q),z\big)^{-1}\beta\big)\!\big)_{\mathrm{B}}\\ &\sim \big(\!\big(e^{\sum_{a=1}^{r'}\bar{p}_a\log q_a/z}\alpha, e^{\sum_{a=1}^{r'}\bar{p}_a\log q_a/z}\beta\big)\!\big)_{\mathrm{B}} = (\!(\alpha,\beta)\!)_{\mathrm{B}} \quad \text{as } q\to 0,\end{aligned}$$

where we used (31), (60) and (99). Since the left-hand side is regular at $z=0$, we know that $(\!(\alpha,\beta)\!)_{\mathrm{B}}$ is regular at $z=0$. Moreover, since $(\cdot,\cdot)_{\mathrm{B}}$ is $\nabla_{z\partial_z}$-flat, we have

$$z\partial_z(\!(\alpha,\beta)\!)_{\mathrm{B}} = \frac{1}{2}(\deg\alpha+\deg\beta-2n)(\!(\alpha,\beta)\!)_{\mathrm{B}} + \frac{1}{z}(\!(\rho\cdot\alpha,\beta)\!)_{\mathrm{B}} - \frac{1}{z}(\!(\alpha,\rho\cdot\beta)\!)_{\mathrm{B}} \tag{101}$$

by the second equation of (12). The last two terms cancel by (99) and so $(\!(\cdot,\cdot)\!)_{\mathrm{B}}$ is of degree $-2n$ when we set $\deg z=2$.

Now we claim that $(\!(\alpha,\beta)\!)_{\mathrm{B}}\in\mathbb{C}$ for $\alpha,\beta\in H^*_{\mathrm{orb}}(\mathcal{X})$. To show the claim, by (99), (100) and the Lefschetz decomposition, it suffices to show that $(\!(\alpha,\omega^k\beta)\!)\in\mathbb{C}$ for primitive classes $\alpha\in H^*(\mathcal{X}_v)$, $\beta\in H^*(\mathcal{X}_{\mathrm{inv}(v)})$ with respect to a Kähler class ω. By the homogeneity (101) of $(\!(\cdot,\cdot)\!)_{\mathrm{B}}$, we have $(\!(\alpha,\omega^k\beta)\!)\in\mathbb{C}z^{k+\frac{1}{2}(\deg\alpha+\deg\beta-2n)}$. By the regularity at $z=0$, this is zero unless $2k+\deg\alpha+\deg\beta\geqslant 2n$. When $2k+\deg\alpha+\deg\beta>2n$, it follows from the Lefschetz decomposition that $\omega^k\alpha=0$ or $\omega^k\beta=0$.

By this claim, one has $(\alpha,\beta)_{\mathrm{B},(\tau(q),z)} = (\!(L(\tau(q),z)^{-1}\alpha, L(\tau(a),z)^{-1}\beta)\!)_{\mathrm{B}} = (\!(\alpha,\beta)\!)_{\mathrm{B}} + O(1/z)$ for $\alpha,\beta\in H^*_{\mathrm{orb}}(\mathcal{X})$. Because $(\alpha,\beta)_{\mathrm{B},(\tau(q),z)}$ is regular at $z=0$, we have $(\alpha,\beta)_{\mathrm{B},(\tau(q),z)} = (\!(\alpha,\beta)\!)_{\mathrm{B}}\in\mathbb{C}$ and this is independent of q and z. Now the ∇-flatness of $(\cdot,\cdot)_{\mathrm{B}}$ gives the Frobenius property

$$\big(\tau_*(\partial_a)\circ\alpha,\beta\big)_{\mathrm{B}} = \big(\alpha,\tau_*(\partial_a)\circ\beta\big)_{\mathrm{B}},\quad \partial_a = q_a(\partial/\partial q_a),$$

where we identify $\tau_*(\partial_a)$ with a section of $(\tau \times \mathrm{id})^* F$ and the subscript $(\tau(q), z)$ is omitted. Since $\tau_*(\partial_a)\circ$ corresponds to the multiplication by p_a in the Batyrev ring (see Proposition 3.10), $\tau_*(\partial_a)$ generates the quantum cohomology over $\mathbf{1}$. Therefore, the pairing $(\cdot,\cdot)_{\mathrm{B}}$ is completely determined by the value $(\mathbf{1}, \gamma)_{\mathrm{B}} \in \mathbb{C}$ for a top-dimensional class $\gamma \in H^{2n}(\mathcal{X})$ and is proportional to $(\cdot,\cdot)_F$.

Finally, we fix the constant ambiguity. Theorem 4.14 implies that the $\Gamma_{\mathbb{R}}$ and Γ_{c} corresponds to the linear functions $\chi(- \otimes \mathcal{O}_{\mathcal{X}}^\vee)$, $\chi(- \otimes \mathcal{O}_{\mathrm{pt}}^\vee)$ on the K-group. (See the proof of Theorem 4.11 in Section 4.3.1.) The pairings match under this correspondence $\sharp(\Gamma_{\mathbb{R}} \cap \Gamma_{\mathrm{c}}) = 1 = \chi(\mathcal{O}_{\mathcal{X}}^\vee \otimes \mathcal{O}_{\mathrm{pt}})$, so the proportionality constant is one.

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